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Estimates on periodic and Dirichlet eigenvalues for the Zakharov–Shabat system

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Abstract. Consider the 2×2 first order system due to Zakharov–Shabat,

$$LY := \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y' + \begin{pmatrix} 0 & \psi_1 \\ \psi_2 & 0 \end{pmatrix} Y = \lambda Y$$

with ψ_1, ψ_2 being complex valued functions of period *one* in the weighted Sobolev space $H^w \equiv H^w_{\mathbb{C}}$. Denote by *spec*$(\psi_1, \psi_2)$ the set of periodic eigenvalues of $L(\psi_1, \psi_2)$ with respect to the interval [0, 2] and by *spec*$_{\mathrm{Dir}}(\psi_1, \psi_2)$ the set of Dirichlet eigenvalues of $L(\psi_1, \psi_2)$ when considered on the interval [0, 1]. It is well known that *spec*(ψ_1, ψ_2) and *spec*$_{\mathrm{Dir}}(\psi_1, \psi_2)$ are discrete.

Theorem. *Assume that w is a weight such that, for some $\delta > 0$, $w_{-\delta}(k) = (1 + |k|)^{-\delta} w(k)$ is a weight as well. Then for any bounded subset $\mathbb{B}$ of 1-periodic elements in $H^w \times H^w$ there exist $N \geqslant 1$ and $M \geqslant 1$ so that for any $|k| \geqslant N$, and $(\psi_1, \psi_2) \in \mathbb{B}$, the set spec$(\psi_1, \psi_2) \cap \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| < \pi/2\}$ contains exactly one isolated pair of eigenvalues $\{\lambda_k^+, \lambda_k^-\}$ and spec$_{\mathrm{Dir}}(\psi_1, \psi_2) \cap \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| < \pi/2\}$ contains a single Dirichlet eigenvalue μ_k. These eigenvalues satisfy the following estimates*

(i) $\sum_{|k| \geqslant N} w(2k)^2 |\lambda_k^+ - \lambda_k^-|^2 \leqslant M$;

(ii) $\sum_{|k| \geqslant N} w(2k)^2 |\frac{(\lambda_k^+ + \lambda_k^-)}{2} - \mu_k|^2 \leqslant M$.

Furthermore spec$(\psi_1, \psi_2) \backslash \{\lambda_k^\pm, |k| \geqslant N\}$ and spec$_{\mathrm{Dir}}(\psi_1, \psi_2) \backslash \{\mu_k \mid |k| \geqslant N\}$ are contained in $\{\lambda \in \mathbb{C} \mid |\lambda| < N\pi - \pi/2\}$ and its cardinality is $4N - 2$, respectively $2N - 1$.

When $\psi_2 = \overline{\psi}_1$ (respectively $\psi_2 = -\overline{\psi}_1$), $L(\psi_1, \psi_2)$ is one of the operators in the Lax pair for the defocusing (resp. focusing) nonlinear Schrödinger equation.

1. Introduction

1.1. Results

Consider the Zakharov–Shabat operator (see [16])

$$L(\psi_1, \psi_2) := \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & \psi_1 \\ \psi_2 & 0 \end{pmatrix},$$

where ψ_1, ψ_2 are 1-*periodic* elements in the weighted Sobolev space $H^w \equiv H^w_{\mathbb{C}}$ of 2-periodic functions

$$H^w := \left\{ f(x) = \sum_{-\infty}^{\infty} \hat{f}(k) \mathrm{e}^{\mathrm{i}\pi k x} \mid \|f\|_w < \infty \right\}$$

with

$$\|f\|_w := \Big(2\sum_{k\in\mathbb{Z}} w(k)^2\big|\hat{f}(k)\big|^2\Big)^{1/2}$$

and $w = (w(k))_{k\in\mathbb{Z}}$ a weight, i.e., a sequence of positive numbers with $w(k) \geqslant 1$, $w(-k) = w(k)$ ($\forall k \in \mathbb{Z}$) and the following submultiplicative property

$$w(k) \leqslant w(k-j)w(j) \quad \forall k,j \in \mathbb{Z}.$$

As an example of such a weight we mention the Sobolev weights $s_n \equiv (s_n(k))_{k\in\mathbb{Z}}$, $s_n(k) := \langle k\rangle^n$, where, for convenience,

$$\langle k\rangle := 1 + |k|,$$

or more generally, the Abel–Sobolev weight $w_{a,b} \equiv (w_{a,b}(k))_{k\in\mathbb{Z}}$

$$w_{a,b}(k) := \langle k\rangle^a \mathrm{e}^{b|k|} \quad (a \geqslant 0, b \geqslant 0).$$

An element $\psi \in H^{w_{a,b}}$ is a complex valued function $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\mathrm{e}^{\mathrm{i}\pi kx}$, which admits an analytic extension $f(x+\mathrm{i}y)$ to the strip $|y| < b/\pi$ such that $f(x+\mathrm{i}\frac{b}{\pi})$ and $f(x-\mathrm{i}\frac{b}{\pi})$ are both in the Sobolev space $H^a_{\mathbb{C}} \equiv H^a(\mathcal{S}^1;\mathbb{C})$. Denote by *spec*$(\psi_1,\psi_2)$ the periodic spectrum of $L(\psi_1,\psi_2)$ when considered on the interval $[0,2]$ and by *spec*$_{\mathrm{Dir}}(\psi_1,\psi_2)$ the Dirichlet spectrum of $L(\psi_1,\psi_2)$ when considered on $[0,1]$. It is well known that both, *spec*(ψ_1,ψ_2) and *spec*$_{\mathrm{Dir}}(\psi_1,\psi_2)$ are discrete.

The main purpose of this paper is to study the asymptotics of the large (in absolute value) eigenvalues in *spec*(ψ_1,ψ_2) and *spec*$_{\mathrm{Dir}}(\psi_1,\psi_2)$ for 1-periodic functions ψ_1,ψ_2 in H^w. To formulate our first result we need to introduce some more notation: we say that w is a δ-weight for $\delta > 0$ if

$$w_*(k) := \langle k\rangle^{-\delta} w(k)$$

is a weight as well. Notice that the Abel–Sobolev weight $w_{a,b}$ is a δ-weight iff $0 < \delta \leqslant a$. Let

$$\delta_* := \delta \wedge \frac{1}{2} \quad \Big(= \inf\Big(\delta, \frac{1}{2}\Big)\Big).$$

Further let

$$\rho_n := \big(\big(\hat{\psi}_2(2n) + \beta_0^+(n)\big)\big(\hat{\psi}_1(-2n) + \beta_0^-(n)\big)\big)^{1/2}$$

with an arbitrary, but fixed choice of the square root and

$$\beta_0^+(n) := \sum_{k,j\neq n} \frac{\hat{\psi}_2(k+n)}{(k-n)\pi}\frac{\hat{\psi}_1(-k-j)}{(j-n)\pi}\hat{\psi}_2(j+n),$$

$$\beta_0^-(n) := \sum_{k,j\neq n} \frac{\hat{\psi}_1(-k-n)}{(k-n)\pi}\frac{\hat{\psi}_2(k+j)}{(j-n)\pi}\hat{\psi}_1(-j-n).$$

The first result concerns the periodic eigenvalues (cf. Section 2).

Theorem 1.1. *Let $M \geqslant 1$, $\delta > 0$ and w a δ-weight. Then there exist constants $1 \leqslant C < \infty$ and $1 \leqslant N < \infty$ so that the following statements hold*:
For any $|n| \geqslant N$ and any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$, the set spec$(\psi_1, \psi_2) \cap \{\lambda \in \mathbb{C} \mid |\lambda - n\pi| < \pi/2\}$ contains exactly one isolated pair of eigenvalues $\{\lambda_k^+, \lambda_k^-\}$. These eigenvalues satisfy

(i) $\sum_{|n|\geqslant N} w(2n)^2 |\lambda_n^+ - \lambda_n^-|^2 \leqslant C$;
(ii) $\sum_{|n|\geqslant N} \langle n\rangle^{3\delta_*} w(2n)^2 \min_\pm |(\lambda_n^+ - \lambda_n^-) \pm 2\rho_n|^2 \leqslant C$;
(iii) *spec$(\psi_1, \psi_2) \backslash \{\lambda_n^\pm \mid |n| \geqslant N\}$ is contained in $\{\lambda \in \mathbb{C} \mid |\lambda| < N\pi - \pi/2\}$ and its cardinality is $4N - 2$.*

Theorem 1.2. *Let $M \geqslant 1$, $\delta > 0$ and w be a δ-weight. Then there exist constants $1 \leqslant C < \infty$ and $N \leqslant N' < \infty$ (with N given by Theorem 1.1) so that the following statements hold*:
For any $|n| \geqslant N'$ and any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$, the set spec$_{\mathrm{Dir}}(\psi_1, \psi_2) \cap \{\lambda \in \mathbb{C} \mid |\lambda - n\pi| < \pi/2\}$ contains exactly one eigenvalue denoted by μ_n. These eigenvalues satisfy:

(i) $\sum_{|n|\geqslant N'} w(2n)^2 |\mu_n - \lambda_n^+|^2 \leqslant C$;
(ii) *spec$_{\mathrm{Dir}}(\psi_1, \psi_2) \backslash \{\mu_n \mid |n| \geqslant N'\}$ is contained in $\{\lambda \in \mathbb{C} \mid |\lambda| < N'\pi - \pi/2\}$ and its cardinality is $2N' - 1$.*

Statement (iii) in Theorem 1.1 and (ii) in Theorem 1.2 are obtained in a standard way. For the convenience of the reader we prove it in Appendix A. In Section 3, we consider the Riesz spaces E_n, i.e., the images of the Riesz projectors associated to $L(\psi_1, \psi_2)$ for a small circle around $n\pi$ with $|n|$ sufficiently large. We analyze the restriction of $L - \lambda_n^+$ to E_n and study the asymptotic properties of eigenfunctions in E_n for $|n| \to \infty$.

1.2. Comments

Operator $L(\psi_1, \psi_2)$: The Zakharov–Shabat operator occurs in the Lax pair representation $\mathrm{d}M_\pm/\mathrm{d}t = [M_\pm, A_\pm]$ of the focusing (*NLS*$_-$) and defocusing (*NLS*$_+$) nonlinear Schrödinger equation

$$\mathrm{i}\partial_t \varphi = -\partial_x^2 \pm 2|\varphi|^2 \varphi,$$

$$M_+ := L(\varphi, \overline{\varphi}), \qquad M_- := L(\varphi, -\overline{\varphi})$$

(whereas the operators $A_\pm$ are rather complicated third order operators, given in [2]). One can show that *spec* $L(\varphi, \overline{\varphi})$ respectively *spec* $L(\varphi, -\overline{\varphi})$ is a complete set of conserved quantities for *NLS*$_+$ respectively *NLS*$_-$. We mention that $L(\psi_1, \psi_2)$ is unitarily equivalent to the *AKNS* operator (see [1,10])

$$L_{AKNS} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} -q & p \\ p & q \end{pmatrix},$$

where

$$\psi_1 := -q + \mathrm{i}p, \qquad \psi_2 = -q - \mathrm{i}p.$$

Hence the selfadjoint operator M_+ corresponds to an operator L_{AKNS} with the functions $q,\ p$ being *real valued*.

Selfadjoint case: We emphasize that Theorems 1.1 and 1.2 do not require $L(\psi_1, \psi_2)$ be selfadjoint. However, in the selfadjoint case, the decay rate of the asymptotics in Theorem 1.1(ii) can be improved from $3\delta_*$ to $4\delta_*$,

$$\sum_{|n|\geqslant N} \langle n\rangle^{4\delta_*} w(2n)^2 \min_{\pm} \left|(\lambda_n^+ - \lambda_n^-) \pm 2\rho_n\right|^2 \leqslant M.$$

(This is proved in Section 2.9.)

L^2**-case:** Theorem 1.1(i) and Theorem 1.2(i) no longer hold for $H^w = L^2$ (i.e., $w(k) = 1\ \forall k \in \mathbb{Z}$) as the number N in Theorem 1.1 cannot be chosen uniformly for 1-periodc functions $\psi_1, \psi_2 \in L^2$ in a L^2-bounded set. This can be easily deduced from the examples considered by Li and McLaughlin [9]: Assume that Theorem 1.1(i) holds for L^2. Given $M > 0$, choose N as in Theorem 1.1 and $\psi_1, \psi_2 \in L^2$ with $\|\psi_j\| \equiv \|\psi_j\|_{L^2} = M$. Define $(\psi_{1,k}, \psi_{2,k}) = (\mathrm{e}^{2\pi \mathrm{i} kx}\psi_1, \mathrm{e}^{-2\pi \mathrm{i} kx}\psi_2)$ $(k \in \mathbb{Z})$. Then $\|\psi_{j,k}\|_{L^2} = \|\psi_j\|_{L^2}$ $(\forall k)$ and, for $n \geqslant N,\ k \geqslant 0$

$$\lambda_{n+k}^{\pm}(\psi_{1,k}, \psi_{2,k}) = \lambda_n^{\pm}(\psi_1, \psi_2) + k\pi$$

which leads for appropriate choices of $\psi_1,\ \psi_2$ to a contradiction. For L selfadjoint, a *local version* of Theorems 1.1 and 1.2 have been established, using different methods, in [3]. Most likely, the analysis presented in this paper can be used to obtain a local version of Theorems 1.1(i) and 1.2(i) for L arbitrary.

Submultiplicative property of weights: Notice that the requirement of a weight to be submultiplicative excludes weights of super-exponential growth $\exp(a|k|^\alpha)$ with $\alpha > 1$. Most likely, the conclusions of Theorems 1.1 and 1.2 do not hold for such weights (cf. [7] for the case of Schrödinger operators).

Boundary conditions: Similarly as in [7] the method for proving Theorem 1.2 can be applied to a whole class of boundary conditions (cf. Section 4 in [7] where this class has been described for the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V$).

Smoothness vs. decay of gap length: For selfadjoint Zakharov–Shabat operators $L(\psi, \overline{\psi})$, Theorem 1.1 has a partial inverse. In this case, the eigenvalues $(\lambda_n^{\pm})_{n\in\mathbb{Z}} = \mathit{spec}\, L(\psi, \overline{\psi})$ are real and can be ordered such that

$$\cdots \leqslant \lambda_{n-1}^+ < \lambda_n^- \leqslant \lambda_n^+ < \lambda_{n+1}^- \leqslant \cdots, \quad \lambda_n^{\pm} = n\pi + \mathrm{o}(1).$$

Given a weight w and $K \geqslant 0$, denote by w_K the weight $w_K(n) := \langle n\rangle^K w(n)$.

Proposition 1.3. *Let w be a δ-weight for some $\delta > 0,\ K \geqslant 0$ and $\varphi \in H^w$. Then $\varphi \in H^{w_K}$ iff*

$$\sum_{n\in\mathbb{Z}} w_K(2n)^2 |\lambda_n^+ - \lambda_n^-|^2 < \infty,$$

where $\lambda_n^{\pm} \equiv \lambda_n^{\pm}(\varphi, \overline{\varphi})$.

In the non selfadjoint case, the smoothness is not characterized by properties of the periodic spectrum alone (cf. [13] for an analysis in the case of Schrödinger operators).

1.3. Method of proof

Typically, asymptotic estimates on the gap's lengths $(\lambda_k^+ - \lambda_k^-)_{k\in\mathbb{Z}}$ of $spec(L(\psi_1,\psi_2))$ are obtained from asymptotic expansions of the eigenvalues $\lambda_k^\pm = k\pi + c_{-1}/k + \cdots$ (cf., e.g., [11]). This approach, however does not allow to obtain the results of Theorems 1.1 and 1.2 for weights with exponential decay such as the Abel–Sobolev weight. The new feature in the proof of our results is to use as in [7] a Lyapunov–Schmidt type decomposition described in detail in Section 2.1.

1.4. Related work

Similar results as the ones presented here for the Zakharov–Shabat operator $L(\psi_1,\psi_2)$ have been obtained previously for the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V$ in [7]. In this paper we document that the same methods, with adjustments, can be applied to L. At first sight this is astonishing, as, unlike in the case of the Schrödinger operator, the distance between adjacent pairs of eigenvalues $(\lambda_n^+,\lambda_n^-)$ and $(\lambda_{n+1}^+,\lambda_{n+1}^-)$ does *not* get unbounded for $|n|\to\infty$, a fact which was used in an essential way in [7]. We explain in Section 2.1 how this problem for L can be overcome.

A weaker version of Theorem 1.1 has been reported in [6] (cf. also [5]).

For Sobolev weights, the asymptotics of the eigenvalues $\lambda_n^\pm$ and hence of the gap length $\gamma_n := \lambda_n^+ - \lambda_n^-$ have been obtained in the selfadjoint case by Marchenko [11] (cf. also [3,4,12,8]). In the non selfadjoint case only a few results have been known so far (see [9,14,15]).

2. Periodic eigenvalues

2.1. Lyapunov–Schmidt decomposition

Consider the Zakharov–Shabat operator

$$L(\psi_1,\psi_2) := \mathrm{i}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix}0 & \psi_1\\ \psi_2 & 0\end{pmatrix},$$

where ψ_1 and ψ_2 are in H^w. For $\psi_1 = \psi_2 = 0$, the periodic eigenvalues are given by $\{\lambda_k^+,\lambda_k^- \mid k\in\mathbb{Z}\}$ with $\lambda_k^+ = \lambda_k^- = k\pi$ and an orthonormal basis of corresponding eigenfunctions in $L^2[0,2]\times L^2[0,2]$ are given by

$$e_k^+(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{\mathrm{i}k\pi x}, \qquad e_k^-(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-\mathrm{i}k\pi x}. \tag{2.1}$$

Considering the multiplication operator $\left(\begin{smallmatrix}0 & \psi_1\\ \psi_2 & 0\end{smallmatrix}\right)$ as a perturbation of the Dirac operator $\mathrm{i}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\frac{\mathrm{d}}{\mathrm{d}x}$ we will see that for k sufficiently large L has a pair of eigenvalues near $k\pi$, isolated from the remaining part of the spectrum of L. Our aim is to obtain an estimate for the distance between the two eigenvalues and to compare the eigenvalues and corresponding eigenfunctions (or root vectors) with the corresponding ones for $\psi_1 = \psi_2 = 0$.

We express the eigenvalue equation

$$LF = \lambda F \tag{2.2}$$

in the basis $e_k^+,\ e_k^-$ ($k \in \mathbb{Z}$) defined in (2.1): Given F in the Sobolev space H^1, write

$$F(x) = \sum_{k\in\mathbb{Z}} \hat{F}_2(k)e_k^+(x) + \hat{F}_1(-k)e_k^-(x) \tag{2.3}$$

and

$$\psi_1(x) = \sum_{k\in\mathbb{Z}} \hat{\psi}_1(k)\mathrm{e}^{\mathrm{i}k\pi x}, \qquad \psi_2(x) = \sum_{k\in\mathbb{Z}} \hat{\psi}_2(k)\mathrm{e}^{\mathrm{i}k\pi x}. \tag{2.4}$$

Substituting (2.3), (2.4) into (2.2) leads to

$$\begin{aligned} LF(x) = {} & \sum_{k\in\mathbb{Z}} k\pi\big(\hat{F}_2(k)e_k^+(x) + \hat{F}_1(-k)e_k^-(x)\big) \\ & + \sum_{k,j\in\mathbb{Z}} \hat{\psi}_1(-k-j)\hat{F}_2(j)e_k^-(x) + \hat{\psi}_2(k+j)\hat{F}_1(-j)e_k^+(x). \end{aligned} \tag{2.5}$$

Hence λ is a periodic eigenvalue of $L(\psi_1, \psi_2)$, when considered on the interval $[0, 2]$, iff there exists $(\hat{F}_1, \hat{F}_2) \in \ell^2 \times \ell^2$ with $(\hat{F}_1, \hat{F}_2) \neq (0, 0)$ such that, for all $k \in \mathbb{Z}$,

$$(k\pi - \lambda)\hat{F}_2(k) + \sum_{j\in\mathbb{Z}} \hat{\psi}_2(k+j)\hat{F}_1(-j) = 0, \tag{2.6}$$

$$(k\pi - \lambda)\hat{F}_1(-k) + \sum_{j\in\mathbb{Z}} \hat{\psi}_1(-k-j)\hat{F}_2(j) = 0. \tag{2.7}$$

Here $\ell^2 \equiv \ell^2(\mathbb{Z};\mathbb{C})$ denotes the Hilbert space of complex valued ℓ^2-sequences $(a(k))_{k\in\mathbb{Z}}$. In order to solve Eqs (2.6), (2.7) we consider a Lyapunov–Schmidt type decomposition. For $n \in \mathbb{Z}$ fixed, we look for eigenvalues near $n\pi$, $\lambda = n\pi + z$, with $|z| \leqslant \pi/2$. The linear system (2.6), (2.7) is then decomposed into a two dimensional system consisting of (2.6), (2.7) with $k = n$, referred to as the $\mathcal{Q}$-equation, and an infinite dimensional system consisting of (2.6), (2.7) with $k \in \mathbb{Z}\backslash\{n\}$, referred to as the $\mathcal{P}$-equation.

First we introduce some more notation. For $K \in \mathbb{Z}$ and a weight w denote by $\ell^2_w(K)$ the complex Hilbert space $\ell^2_w(K) \equiv \ell^2_w(K, \mathbb{C})$,

$$\ell^2_w(K) := \big\{\big(a(k)\big)_{k\in K} \mid \|a\|_w < \infty\big\},$$

where $\|a\|_w = (a, a)_w^{1/2}$ and, for $a, b \in \ell^2_w$,

$$(a, b)_w := \sum_{k\in K} w(k)^2\overline{a(k)}b(k).$$

Most frequently, we will use for K the set $\mathbb{Z}$ or $\mathbb{Z}\backslash n \equiv \mathbb{Z}\backslash\{n\}$. If necessary for clarity, we write a_K for a sequence $(a(k))_{k\in K} \in \ell^2_w(K)$.

For a linear operator $A : \ell^2_{w_1}(K_1) \to \ell^2_{w_2}(K_2)$ we denote by $A(k,j)$ its matrix elements,

$$(Aa)(k) := \sum_{j \in K_1} A(k,j)a(j) \quad (k \in K_2).$$

Further we introduce the shift operator S and an involution operator $\mathcal{J}$

$$S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (Sa)(k) := a(k+1) \quad \forall k \in \mathbb{Z},$$
$$J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (\mathcal{J}a)(k) := a(-k) \quad \forall k \in \mathbb{Z}.$$

The restriction of S to $\ell^2_w(K)$ with values in $\ell^2_{S^n w}(K)$ is again denoted by S and $S^n := S \circ \cdots \circ S$ denotes the n-th iterate of S. Notice that

$$\|S^n a\|^2_{\ell^2_{S^n w}(K)} = \sum_{k \in K} w(k+n)^2 \big|a(k+n)\big|^2 \leqslant \|a\|^2_{\ell^2_w(\mathbb{Z})}.$$

For $(\hat{F}_2, \hat{F}_1) \in \ell^2 \times \ell^2$, write

$$\hat{F}_2 = \big(x^F, \breve{F}_2\big), \quad x^F := \hat{F}_2(n); \quad \breve{F}_2 := \big(\hat{F}_2(k)\big)_{k \in \mathbb{Z}\setminus n},$$
$$\hat{F}_1 = \big(y^F, J\breve{F}_1\big), \quad y^F := \hat{F}_1(-n); \quad \breve{F}_1 := \big(\hat{F}_1(k)\big)_{k \in \mathbb{Z}\setminus n}.$$

Using the above introduced notation, Eqs (2.6), (2.7) read as follows:

$$-zx^F + \hat{\psi}_2(2n)y^F + \big\langle S^n\hat{\psi}_2, J\breve{F}_1\big\rangle = 0, \tag{2.8}$$
$$\hat{\psi}_1(-2n)x^F - zy^F + \big\langle S^n J\hat{\psi}_1, \breve{F}_2\big\rangle = 0 \tag{2.9}$$

and

$$\begin{pmatrix} y^F(S^n\hat{\psi}_2)_{\mathbb{Z}\setminus n} \\ x^F(S^n J\hat{\psi}_1)_{\mathbb{Z}\setminus n} \end{pmatrix} + (A_n - z)\begin{pmatrix} \breve{F}_2 \\ J\breve{F}_1 \end{pmatrix} = 0. \tag{2.10}$$

Eqs (2.8), (2.9) together form the $\mathcal{Q}$-equation and (2.10) is the $\mathcal{P}$-equation. The operator A_n is given by

$$A_n = \begin{pmatrix} ((k-n)\pi\delta_{kj})_{k,j\in\mathbb{Z}\setminus n} & (\hat{\psi}_2(k+j))_{k,j\in\mathbb{Z}\setminus n} \\ ((J\hat{\psi}_1(k+j))_{k,j\in\mathbb{Z}\setminus n} & ((k-n)\pi\delta_{kj})_{k,j\in\mathbb{Z}\setminus n} \end{pmatrix}$$

and $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_{\mathbb{Z}\setminus n}$ is defined by (no complex conjugation)

$$\left\langle \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \begin{pmatrix} c_n \\ d_n \end{pmatrix} \right\rangle := \sum_{k\in\mathbb{Z}\setminus n} \big(a_n(k)c_n(k) + b_n(k)d_n(k)\big).$$

For $\psi_1 = \psi_2 = 0$ and $|z| \leqslant \pi/2$, the operator $z - A_n$ is invertible as $(k\pi - (n\pi - z)) \neq 0$ for $k \neq n$. By a perturbation argument we will show that $(z - A_n)$ can be inverted for $|z| \leqslant \pi/2$ and $|n|$ sufficiently large which then allows to solve the $\mathcal{P}$-equation (2.10) for $(\breve{F}_2, J\breve{F}_1)$ for any $x^F, y^F \in \mathbb{C}$. This solution is substituted into (2.8), (2.9) which leads to a homogeneous linear system of two equations for x^F and y^F with coefficients which depend on the parameter z. Hence $\lambda = n\pi + z$ is a periodic eigenvalue of $L(\psi_1, \psi_2)$ iff the corresponding determinant is equal to 0. The nature of the latter equation allows to obtain asymptotics for the difference $\lambda_n^+ - \lambda_n^-$ without having to compute the asymptotics of λ_n^+ and λ_n^- (cf. Sections 2.6 and 2.7).

2.2. *P-equation*

Let us first introduce some more notation. Denote by $\varDelta_n$ the diagonal part of A_n

$$\varDelta_n := \begin{pmatrix} D_n & 0 \\ 0 & D_n \end{pmatrix}, \quad D_n := \big((k-n)\pi\delta_{kj}\big)_{k,j\in\mathbb{Z}\setminus n}$$

and set

$$B_n := A_n - \varDelta_n.$$

Notice that for $|z| \leqslant \pi/2$, $(z - \varDelta_n)^{-1}$ is invertible. Hence we may introduce

$$T_n \equiv T_{n,z} := B_n(z-\varDelta_n)^{-1} = \begin{pmatrix} 0 & R_n^{(2)} \\ R_n^{(1)} & 0 \end{pmatrix}, \tag{2.11}$$

where $R_n^{(j)} \equiv R_{n,z}^{(j)} : \ell^2(\mathbb{Z}\setminus n) \to \ell^2(\mathbb{Z}\setminus n)$ are defined by

$$R_n^{(1)}(a) := J\big(\hat{\psi}_1 * (z-D_n)^{-1}\big)a, \qquad R_n^{(2)}(a) := \hat{\psi}_2 * J(z-D_n)^{-1}a. \tag{2.12}$$

$R_n^{(1)}$ and $R_n^{(2)}$ have the following matrix representations

$$R_n^{(1)}(k,j) := \frac{\hat{\psi}_1(-k-j)}{z-(j-n)\pi}, \qquad R_n^{(2)}(k,j) := \frac{\hat{\psi}_2(k+j)}{z-(j-n)\pi} \quad (k,j\in\mathbb{Z}\setminus n). \tag{2.13}$$

Formally, for any $x^F, y^F \in \mathbb{C}$, the $\mathcal{P}$-equation (2.10) can be solved

$$\begin{pmatrix} \breve{F}_2 \\ J\breve{F}_1 \end{pmatrix} = (z-A_n)^{-1}\begin{pmatrix} y^F S^n \hat{\psi}_2 \\ x^F S^n J\hat{\psi}_1 \end{pmatrix}$$

with

$$(z-A_n)^{-1} = (z-\varDelta_n)^{-1}(Id - T_n)^{-1}. \tag{2.14}$$

To justify the formal considerations above it is to show that $Id - T_n$ is invertible. Unfortunately, the norm $\|T_n\|$ of T_n in $\mathcal{L}(\ell^2_{S^n w})$ (with $\ell^2_{S^n w} \equiv \ell^2_{S^n w}(\mathbb{Z}\setminus n; \mathbb{C}^2)$) does not become small as $|n| \to \infty$. However, it turns out that, assuming an additional condition on the weight, the norm of T_n^2 is small for $|n| \to \infty$. The invertibility of $Id - T_n$ then follows from the identity

$$Id = (Id - T_n) \circ (Id + T_n)\big(Id - T_n^2\big)^{-1}. \tag{2.15}$$

Given $\varphi \in H^w$, denote by Φ_n the operator in $\mathcal{L}(\ell^2)$ (with $\ell^2 \equiv \ell^2(\mathbb{Z};\mathbb{C})$) defined by ($n \in \mathbb{Z}; a \in \ell^2(\mathbb{Z};\mathbb{C})$)

$$(\Phi_n a)(k) := \sum_{j\in\mathbb{Z}} \frac{\hat\varphi(k+j)}{\langle n-j\rangle} a(j) \quad (\forall k \in \mathbb{Z}),$$

where $\langle k\rangle = 1 + |k|$.

Recall that a weight w is called a δ-weight ($\delta \geqslant 0$) if $w_{-\delta}(k) := \langle k\rangle^{-\delta} w(k)$ is a weight. For convenience we denote the weight $w_{-\delta}$ by w_*. The two key lemmas for proving that $\lim_{n\to\infty} \|T_n^2\| = 0$ are the following ones:

Lemma 2.1. *Let w be a δ-weight with $0 \leqslant \delta < 1/2$ and $n \in \mathbb{Z}$. Then there exists $C = C(\delta)$ such that*

$$\|\Phi_n\|_{\mathcal{L}(\ell^2_{S^{-n}w_*};\ell^2_{S^n w})} \leqslant C\|\varphi\|_w.$$

Proof. For $a \in \ell^2_{S^{-n}w_*}$ and $b \in \ell^2_{S^n w}$,

$$\begin{aligned}\left|(b,\Phi_n a)_{S^n w}\right| \leqslant{} & \sum_{j,k} w(k+n)\left|b(k)\right| w_*(j-n)\left|a(j)\right| w(k+j)\left|\hat\varphi(k+j)\right| \\ & \times \frac{w(k+n)}{w_*(j-n)w(k+j)} \frac{4}{\langle n-j\rangle}.\end{aligned}$$

Using that w is submultiplicative, one gets

$$\frac{w(k+n)}{w_*(j-n)w(k+j)} \leqslant \frac{w(n-j)}{w_*(j-n)} = \langle j-n\rangle^\delta \leqslant \left(4\left|n-j+\frac{1}{2}\right|\right)^\delta \leqslant 2\left|n-j+\frac{1}{2}\right|^\delta$$

and hence, by the Cauchy–Schwartz inequality

$$\begin{aligned}\left|(b,\Phi_n a)_{S^n w}\right| &\leqslant \|b\|_{S^n w}\|a\|_{S^{-n}w_*}\left(\sum_{k,j} \frac{4|\hat\varphi(k+j)|^2 w(k+j)^2}{\langle n-j\rangle^{2(1-\delta)}}\right)^{1/2} \\ &\leqslant C\|b\|_{S^n w}\|a\|_{S^{-n}w_*}\|\varphi\|_w\end{aligned}$$

with $C \equiv C(\delta) := (\sum_k 4/\langle k\rangle^{2-2\delta})^{1/2} < \infty$ as $\delta < 1/2$. □

Lemma 2.2. *Let $\delta \geqslant 0$, w be a δ-weight and $n \in \mathbb{Z}$. Then there exists $C > 0$, independent of δ, such that*

$$\|\Phi_n\|_{\mathcal{L}(\ell^2_{S^n w};\ell^2_{S^{-n}w_*})} \leqslant C\frac{\|\varphi\|_{w_*}}{\langle n\rangle^{\delta\wedge 1}},$$

where as usual $\delta \wedge 1 = \min(1,\delta)$.

Proof. For $a \in \ell^2_{S^n w}$ and $b \in \ell^2_{S^{-n} w_*}$,

$$\left|(b, \Phi_n a)_{S^{-n} w_*}\right| \leqslant \sum_{k,j} w_*(k-n)\left|b(k)\right| w(j+n)\left|a(j)\right| w_*(k+j)\left|\hat{\varphi}(k+j)\right| \\ \times \frac{w_*(k-n)}{w(j+n) w_*(k+j)} \frac{4}{\langle n-j\rangle}.$$

As w_* submultiplicative and symmetric,

$$w_*(k-n) \leqslant w_*(k+j) w_*(j+n)$$

which leads to (use definition of w_*)

$$\left|(b, \Phi_n a)_{S^{-n} w_*}\right| \leqslant \|b\|_{S^{-n} w_*} \|a\|_{S^n w} \|\hat{\varphi}\|_{w_*} \left(\sum_j \frac{4}{\langle j+n\rangle^{2\delta}} \frac{4}{\langle j-n\rangle^2}\right)^{1/2}.$$

The claimed estimate then follows from the following elementary estimate

$$\left(\sum_j \frac{1}{\langle j+n\rangle^{2\delta}} \frac{1}{\langle j-n\rangle^2}\right)^{1/2} \leqslant C \frac{1}{\langle n\rangle^{\delta \wedge 1}}$$

for some C, independent of δ. $\square$

As an application of Lemmas 2.1 and 2.2 we obtain estimates for the norms of $R_n^{(j)}$, T_n and T_n^2. By definition

$$T_n^2 = \begin{pmatrix} 0 & R_n^{(2)} \\ R_n^{(1)} & 0 \end{pmatrix}^2 = \begin{pmatrix} R_n^{(2)} R_n^{(1)} & 0 \\ 0 & R_n^{(1)} R_n^{(2)} \end{pmatrix} \tag{2.16}$$

and it is useful to introduce the operators

$$P_n := R_n^{(2)} R_n^{(1)}, \qquad Q_n := R_n^{(1)} R_n^{(2)}. \tag{2.17}$$

To make notation easier we write $\ell^2_{S^{\pm n} w}$ for both, $\ell^2_{S^{\pm n} w}(\mathbb{Z}\backslash n; \mathbb{C})$ and $\ell^2_{S^{\pm n} w}(\mathbb{Z}\backslash n; \mathbb{C}^2)$.

Corollary 2.3. *Let $\delta \geqslant 0$, $M \geqslant 1$ and w be a δ-weight. Then, for any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$ $(j = 1, 2)$, the following statements hold*:

(i) *If $0 \leqslant \delta < 1/2$, there exists $C \equiv C(\delta) > 0$ so that for $1 \leqslant j \leqslant 2$, $n \in \mathbb{Z}$, and $|z| \leqslant \pi/2$,*

$$\|R_n^{(j)}\|_{\mathcal{L}(\ell^2_{S^{-n} w_*}; \ell^2_{S^n w})} \leqslant CM, \qquad \|T_n\|_{\mathcal{L}(\ell^2_{S^{-n} w_*}; \ell^2_{S^n w})} \leqslant CM.$$

(ii) *If $\delta \geqslant 0$, there exists $C > 0$ such that for $1 \leqslant j \leqslant 2$, $n \in \mathbb{Z}$, and $|z| \leqslant \pi/2$,*

$$\|R_n^{(j)}\|_{\mathcal{L}(\ell^2_{S^n w}, \ell^2_{S^{-n} w_*})} \leqslant \frac{CM}{\langle n\rangle^{\delta \wedge 1}}, \qquad \|T_n\|_{\mathcal{L}(\ell^2_{S^n w}, \ell^2_{S^{-n} w_*})} \leqslant \frac{CM}{\langle n\rangle^{\delta \wedge 1}}.$$

(iii) *If* $0 \leqslant \delta < 1/2$*, then there exists* $C \equiv C(\delta)$ *so that for* $n \in \mathbb{Z}$ *and* $|z| \leqslant \pi/2$*,*

$$\|P_n\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant \frac{CM^2}{\langle n\rangle^\delta}, \qquad \|Q_n\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant \frac{CM^2}{\langle n\rangle^\delta},$$
$$\|P_n\|_{\mathcal{L}(\ell^2_{S^{-n} w_*})} \leqslant \frac{CM^2}{\langle n\rangle^\delta}, \qquad \|Q_n\|_{\mathcal{L}(\ell^2_{S^{-n} w_*})} \leqslant \frac{CM^2}{\langle n\rangle^\delta}.$$

Proof. The claimed estimates for $R_n^{(j)}$ $(j = 1,2)$ follow from Lemmas 2.1 and 2.2. As $T_n = \left(\begin{smallmatrix} 0 & R_n^{(2)} \\ R_n^{(1)} & 0 \end{smallmatrix}\right)$, these estimates then imply the ones for T_n. The estimates in (iii) are obtained by combining the estimates in (i) and (ii) for $R_n^{(j)}$. □

Under the assumptions of Corollary 2.3 define, for $0 < \delta < 1/2$ and $M \geqslant 1$,

$$N_0 \equiv N_0(\delta, M, w) := \max\bigl(1, (2CM^2)^{1/\delta}\bigr) \tag{2.18}$$

with C given as in Corollary 2.3(iii).

Proposition 2.4. *Let* $0 < \delta < 1/2$*,* $M \geqslant 1$ *and* w *be a* δ*-weight. Then, for any* 1*-periodic functions* $\psi_1, \psi_2 \in H^w$ *with* $\|\psi_j\|_w \leqslant M$*,* $|n| \geqslant N_0$ *and* $|z| \leqslant \pi/2$*,*

(i) $$\|P_n\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant \frac{1}{2}, \qquad \|Q_n\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant \frac{1}{2};$$

(ii) *Id* $- P_n$ *and Id* $- Q_n$ *are invertible and*

$$\|(Id - P_n)^{-1}\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant 2, \qquad \|(Id - Q_n)^{-1}\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant 2.$$

(iii) *Id* $- T_n^2$ *is invertible and*

$$\|T_n^2\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant \frac{1}{2}, \qquad \|(Id - T_n^2)^{-1}\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant 2.$$

(iv) *Statements* (i)–(iii) *remain true if one replaces the weight* $S^n w$ *by* $S^{-n} w_*$*.*

Proof. (i) By Corollary 2.3(iii) P_n satisfies the estimate (as $0 < \delta < 1/2$)

$$\|P_n\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant \frac{CM^2}{\langle n\rangle^\delta}.$$

Hence for $|n| \geqslant N_0$

$$\|P_n\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant \frac{CM^2}{\langle N_0\rangle^\delta} \leqslant \frac{1}{2}.$$

Similarly, one obtains $\|Q_n\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant 1/2$.

(ii) follows immediately from (i) and (iii) follows from (i), (ii) and the identity $T_n^2 = \left(\begin{smallmatrix} P_n & 0 \\ 0 & Q_n \end{smallmatrix}\right)$. Finally, statements (i)–(iii) for the weight $S^{-n}w_*$ are proved in a similar way as for $S^n w$. $\square$

Summarizing the results obtained in this section, we obtain, with $\|\cdot\| \equiv \|\cdot\|_{\mathcal{L}(\ell^2_{S^n w}(\mathbb{Z}\backslash n;\mathbb{C}^2))}$:

Corollary 2.5. *Let* $0 < \delta < 1/2,\ M \geqslant 1$ *and* w *be a* δ*-weight. Then there exists* $C > 0$ *such that, for any* 1*-periodic functions* $\psi_1, \psi_2 \in H^w$ *with* $\|\psi_j\|_w \leqslant M$ $(j = 1, 2)$, $|n| \geqslant N_0$ *and* $|z| \leqslant \pi/2$:

(i) $\|T_n\| \leqslant C$;
(ii) $Id - T_n$ *is invertible in* $\mathcal{L}(\ell^2_{S^n w}(\mathbb{Z}\backslash n;\mathbb{C}^2))$ *and* $\|(Id - T_n)^{-1}\| \leqslant C$;
(iii) $z - A_n$ *is invertible in* $\mathcal{L}(\ell^2_{S^n w}(\mathbb{Z}\backslash n;\mathbb{C}^2))$ *and* $\|(z - A_n)^{-1}\| \leqslant C$.

Proof. (i) Recall that $T = \left(\begin{smallmatrix} 0 & R_n^{(2)} \\ R_n^{(1)} & 0 \end{smallmatrix}\right)$. By standard convolution estimates, there exists an absolute constant $C > 0$ so that for $n \in \mathbb{Z}$ and $|z| \leqslant \pi/2,\ \|\psi_j\|_w \leqslant M$

$$\|T_n\| \leqslant CM.$$

Therefore (ii) and (iii) follow immediately from (2.14), (2.15) and Proposition 2.4. $\square$

2.3. Q-equation

Using the notations introduced in Section 2.2, we have for $|n| \geqslant N_0$ and $|z| \leqslant \pi/2$

$$(z - A_n)^{-1} = \begin{pmatrix} (z - D_n)^{-1}(Id - P_n)^{-1} & (z - D_n)^{-1}R_n^{(2)}(Id - Q_n)^{-1} \\ (z - D_n)^{-1}R_n^{(1)}(Id - P_n)^{-1} & (z - D_n)^{-1}(Id - Q_n)^{-1} \end{pmatrix}.$$

Hence the P-equation (2.10) leads to the following formulas

$$\breve{F}_2 = y^F(z - D_n)^{-1}(Id - P_n)^{-1}S^n\hat{\psi}_2 + x^F(z - D_n)^{-1}R_n^{(2)}(Id - Q_n)^{-1}S^n J\hat{\psi}_1, \tag{2.19}$$

$$J\breve{F}_1 = y^F(z - D_n)^{-1}R_n^{(1)}(Id - P_n)^{-1}S^n\hat{\psi}_2 + x^F(z - D_n)^{-1}(Id - Q_n)^{-1}S^n J\hat{\psi}_1. \tag{2.20}$$

These solutions are substituted into the Q-equations (2.8), (2.9) to obtain for $|z| \leqslant \pi/2,\ |n| \geqslant n_0$ the following homogeneous system:

$$\big(-z + \alpha^+(n,z)\big)x^F + \big(\hat{\psi}_2(2n) + \beta^+(n,z)\big)y^F = 0, \tag{2.21}$$

$$\big(\hat{\psi}_1(-2n) + \beta^-(n,z)\big)x^F + \big(-z + \alpha^-(n,z)\big)y^F = 0, \tag{2.22}$$

where

$$\alpha^+(n,z) := \big\langle S^n\hat{\psi}_2, (z - D_n)^{-1}(Id - Q_n)^{-1}S^n J\hat{\psi}_1\big\rangle, \tag{2.23}$$

$$\beta^+(n,z) := \big\langle S^n\hat{\psi}_2, (z - D_n)^{-1}R_n^{(1)}(Id - P_n)^{-1}S^n\hat{\psi}_2\big\rangle, \tag{2.24}$$

$$\alpha^-(n,z) := \big\langle S^n J\hat{\psi}_1, (z - D_n)^{-1}(Id - P_n)^{-1}S^n\hat{\psi}_2\big\rangle, \tag{2.25}$$

$$\beta^-(n,z) := \big\langle S^n J\hat{\psi}_1, (z - D_n)^{-1}R_n^{(2)}(Id - Q_n)^{-1}S^n J\hat{\psi}_1\big\rangle. \tag{2.26}$$

Notice that $\alpha^{\pm}(n,z)$ and $\beta^{\pm}(n,z)$ are analytic for $|z| < \pi/2$ as $R_n^{(j)}$, P_n and Q_n are analytic for $|z| < \pi/2$. An important simplification of Eqs (2.21), (2.22) results from the following observation:

Lemma 2.6. *For $|z| \leqslant \pi/2$ and $|n| \geqslant N_0$,*

$$\alpha^+(n,z) = \alpha^-(n,z).$$

Proof. In view of (2.23) and (2.25) it is to show that

$$(z - D_n)^{-1}(Id - Q_n)^{-1} = \big((Id - P_n)^{-1}\big)^t (z - D_n)^{-1}, \tag{2.27}$$

where A^t denotes the transpose of A,

$$\big(A^t\big)(k,j) := A(j,k) \quad (\text{no complex conjugation}).$$

Eq. (2.27) can be reformulated,

$$\big((Id - Q_n)(z - D_n)\big)^{-1} = \big((z - D_n)(Id - P_n^t)\big)^{-1},$$

which holds iff

$$Q_n(z - D_n) = \big(P_n(z - D_n)\big)^t. \tag{2.28}$$

The identity (2.28) follows easily from

$$\big(Q_n(z - D_n)\big)(j,k) = \big(R_n^{(1)} R_n^{(2)}\big)(j,k)\big(z - (k-n)\pi\big) = \sum_{\ell} \frac{\hat{\psi}_1(-j-\ell)}{z - (\ell - n)\pi} \cdot \hat{\psi}_2(\ell + k)$$

and

$$\begin{aligned} \big(P_n(z - D_n)\big)^t(j,k) &= \big(P_n(z - D_n)\big)(k,j) = \big(R_n^{(2)} R_n^{(1)}\big)(k,j)\big(z - (j-n)\pi\big) \\ &= \sum_{\ell} \frac{\hat{\psi}_2(k+\ell)}{z - (\ell - n)\pi} \hat{\psi}_1(-\ell - j). \qquad \square \end{aligned}$$

In view of Lemma 2.6 we write

$$\alpha(n,z) := \alpha^+(n,z) \quad \big(= \alpha^-(n,z)\big). \tag{2.29}$$

In subsequent sections we estimate the coefficients $\alpha(n,z)$, $\beta^+(n,z)$ and $\beta^-(n,z)$.

2.4. Estimates for $\alpha(n,z)$

Lemma 2.7. *Let $0 < \delta \leqslant 1/2$, $M \geqslant 1$ and w be a δ-weight. Then, for any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$, $|n| \geqslant N_0$ ($N_0 \equiv N_0(\delta, M)$ given by (2.18)) and $|z| \leqslant \pi/2$*

$$|\alpha(n,z)| \leqslant \frac{4M^2}{\langle n\rangle^{2\delta}}.$$

Proof. Write $\alpha(n,z) = \langle S^n\hat{\psi}_2, (z-D_n)^{-1}a\rangle$ with $a := (Id-Q_n)^{-1}S^nJ\hat{\psi}_1 \in \ell^2_{S^n w}$. By Proposition 2.4,

$$\|a\|_{S^n w} \leqslant 2\|\hat{\psi}_1\|_w \leqslant 2M.$$

Hence

$$\langle n\rangle^{2\delta}|\alpha(n,z)| \leqslant \sum_{k\neq n} \frac{\langle n\rangle^{2\delta}}{\langle n-k\rangle}|\hat{\psi}_2(k+n)||a(k)| \leqslant \sum_{|k+n|<|n|} \frac{\langle n\rangle^{2\delta}}{\langle n\rangle}|\hat{\psi}_2(k+n)||a(k)|$$
$$+ \sum_{|k+n|\geqslant|n|} \frac{\langle n\rangle^{2\delta}}{w(k+n)^2} w(k+n)|\hat{\psi}_2(k+n)|w(k+n)|a(k)| \leqslant 2\|\hat{\psi}_2\|_w\|a\|_{S^n w},$$

where we used that $2\delta \leqslant 1$ and $w(k+n) = w_*(k+n)\langle k+n\rangle^\delta \geqslant \langle n\rangle^\delta$ for $|k+n| \geqslant |n|$. $\square$

2.5. Estimates for $\beta^\pm(n,z)$

In this section we provide estimates for $\beta^\pm(n,z)$. The $\beta^\pm(n,z)$ – they turn out to be quite small – determine the asymptotics of the sequence of gap lengths given in Theorem 1.1. As $\beta^+(n,z)$ (cf. (2.24)) and $\beta^-(n,z)$ (cf. (2.26)) are analyzed in a similar fashion we focus on the estimate for $\beta^+(n,z)$. Writing $(Id - P_n)^{-1} = \sum_{k=0}^\infty P_n^k$ we obtain for $\beta^+(n,z)$ the following convergent series

$$\beta^+(n,z) = \sum_{k=0}^\infty \beta_k(n,z), \tag{2.30}$$

where

$$\beta_k(n,z) := \langle S^n\hat{\psi}_2, (z-D_n)^{-1}R_n^{(1)}P_n^k S^n\hat{\psi}_2\rangle. \tag{2.31}$$

The convergence of series (2.30) follows from $\|P_n\|_{\mathcal{L}(\ell^2_{S^{\pm n}w})} \leqslant 1/2$ ($|n| \geqslant N_0$, Proposition 2.4). We begin by analyzing $\beta_k(n,z)$.

With $R_n^{(1)}$ defined by (cf. (2.12))

$$(R_n^{(1)}a)(j) = J(\hat{\psi}_1 * (z-D_n)^{-1}a)(j) = \sum_{\ell\neq n} \frac{(J\hat{\psi}_1)(j+\ell)a(\ell)}{z-(\ell-n)\pi}$$

and $\inf_{|z|\leqslant\pi/2}|z-(\ell-n)\pi| \geqslant \frac{1}{2}\langle \ell-n\rangle$ (for any $\ell\neq n$) we get

$$\left|\left(R_n^{(1)}a\right)(j)\right| \leqslant 2\sum_{\ell}\frac{|J\hat{\psi}_1(j+\ell)|}{\langle \ell-n\rangle}|a(\ell)|$$

which leads to

$$\left|\beta_k(n,z)\right| \leqslant 4\sum_j \frac{|\hat{\psi}_2(n+j)|}{\langle j-n\rangle}\sum_{\ell}\frac{|J\hat{\psi}_1(j+\ell)|}{\langle \ell-n\rangle}\left|\left(S^{-n}P_n^kS^n\hat{\psi}_2\right)(\ell+n)\right|. \tag{2.32}$$

Given three nonnegative sequences (i.e., sequences of nonnegative numbers), a, b, d in $\ell^2(\mathbb{Z})$ we define, for any $n\in\mathbb{Z}$, the sequence $\Psi_n \equiv \Psi_n(a,b,d)$ by

$$\Psi_n(k+n) := \sum_j \frac{a(k+j)}{\langle j-n\rangle}\sum_{\ell}\frac{b(j+\ell)}{\langle \ell-n\rangle}d(\ell+n).$$

Then Ψ_n is a nonnegative sequence in $\ell^2(\mathbb{Z})$ and can be used to rewrite (2.32). Introduce, for $|n|\geqslant N_0$ and $|z|\leqslant \pi/2$,

$$\eta_{n,0} := 4\Psi_n\left(|\hat{\psi}_2|,|J\hat{\psi}_1|,|\hat{\psi}_2|\right) \tag{2.33}$$

and, for $k\geqslant 0$,

$$\eta_{n,k+1} := 4\Psi_n\left(|\hat{\psi}_2|,|J\hat{\psi}_1|,\eta_{n,k}\right), \tag{2.34}$$

where, for any $a=(a(j))_{j\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$, we denote by $|a|$ the sequence $(|a(j)|)_{j\in\mathbb{Z}}$. As for any $|z|\leqslant\pi/2$,

$$\begin{aligned}
&\left|\left(S^{-n}P_nS^na\right)(k+n)\right| \\
&\quad= \left|\left(P_nS^na\right)(k)\right| = \left|\left(R_n^{(2)}R_n^{(1)}S^na\right)(k)\right| = \left|\left(\hat{\psi}_2 * \left(J(z-D_n)^{-1}\left(R_n^{(1)}S^{-n}a\right)\right)\right)(k)\right| \\
&\quad\leqslant 4\sum_j\left|\hat{\psi}_2(k+j)\right|\frac{1}{\langle j-n\rangle}\sum_{\ell}\frac{|J\hat{\psi}_1(j+\ell)|}{\langle \ell-n\rangle}\left|a(\ell+n)\right| = 4\Psi_n\left(|\hat{\psi}_2|,|J\hat{\psi}_1|,|a|\right)
\end{aligned}$$

it follows, by an induction argument, from (2.32) and

$$S^{-n}P_n^kS^n\hat{\psi}_2 = \left(S^{-n}P_nS^n\right)\left(S^{-n}P_n^{k-1}S^n\psi_2\right)$$

that, for any $k\geqslant 0$,

$$\sup_{|z|\leqslant\pi/2}\left|\beta_k(n,z)\right| \leqslant \eta_{n,k}(2n). \tag{2.35}$$

To estimate $\eta_{n,k}(2n)$, we need the following auxilary lemma concerning the operator Ψ_n. For $\delta>0$ and w be a δ-weight, define

$$\delta_* = \delta\wedge\frac{1}{2}.$$

Lemma 2.8. *Let w a δ-weight, and, for any $n \in \mathbb{Z}$, d_n a positive sequence in ℓ_w^2 so that*

$$\langle n\rangle^{\alpha} d_n(j) \leqslant d(j) \quad \forall n, j \in \mathbb{Z}$$

for some $\alpha \geqslant 0$ and some positive sequence d in ℓ_w^2. Then there exist $C \equiv C_{\delta_*}$, only depending on δ_*, and $e \in \ell_w^2$ so that for any positive sequences $a, b \in \ell_w^2$,

(i) $$\sum_{n\in\mathbb{Z}} \langle n\rangle^{2(2\delta_*+\alpha)} w(2n)^2 \big(4\Psi_n(a,b,d_n)(2n)\big)^2 \leqslant C\|a\|_w\|b\|_w\|d\|_w;$$

(ii) *for any $n, j \in \mathbb{Z}$,*

$$\langle n\rangle^{\alpha+\delta_*} 4\Psi_n(a,b,d_n)(j) \leqslant e(j), \qquad \|e\|_w \leqslant C\|a\|_w\|b\|_w\|d\|_w.$$

Proof. Cf. Appendix B. $\square$

From Lemma 2.8 we obtain, in view of the definition (2.33), (2.34) and the estimate (2.35) the following

Corollary 2.9. *Let $M \geqslant 1$ and w be a δ-weight. Then for any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$ $(j = 1, 2)$*

(i) *for $k \geqslant 0$*

$$\sum_{|n|\geqslant N_0} \langle n\rangle^{2(2+k)\delta_*} w(2n)^2 \sup_{|z|\leqslant\pi/2} \big|\beta_k(n,z)\big|^2 \leqslant C^{k+1} M^{2k+3},$$

where $1 \leqslant C \equiv C_\delta < \infty$ is given by Lemma 2.8

(ii) $$\sum_{|n|\geqslant N_0} \langle n\rangle^{6\delta_*} w(2n)^2 \sup_{|z|\leqslant\pi/2} \big|\tilde\beta(n,z)\big|^2 \leqslant C',$$

where $\tilde\beta(n,z) := \sum_{k\geqslant1} \beta_k(n,z)$ and $1 \leqslant C' < \infty$ is a constant depending only on M and δ.

Proof. We apply Lemma 2.8 to each of the β_k's in an inductive fashion to obtain (i). Statement (ii) then follows from (i) by the Cauchy–Schwartz inequality. $\square$

To simplify further the asymptotics of β write $\beta_0(n,z) \equiv \beta_0^+(n,z) = \beta_0^+(n) + z\beta_\#^+(n,z)$, where

$$\beta_0^\pm(n) := \beta_0^\pm(n,0), \qquad \beta_\#^\pm(n,z) := \int_0^1 \partial_z\beta_0^\pm(n,tz)\,\mathrm{d}t.$$

As $z \mapsto \beta_\#^\pm(n,z)$ are analytic functions in $\{|z| < \pi/2\}$, one deduces by Cauchy's formula

$$\sup_{|z|\leqslant\pi/4} \big|\beta_\#^\pm(n,z)\big| \leqslant \frac{4}{\pi} \sup_{|z|\leqslant\pi/2} \big|\beta_0^\pm(n,z)\big|. \tag{2.36}$$

Summarizing our results of this section gives the following

Proposition 2.10. *Let $\delta > 0$, $M \geqslant 1$, $1 \leqslant A < \infty$ and w be a δ-weight. Then there exists $C > 0$ so that for any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$ $(j = 1, 2)$,*

(i) $$\sum_{|n| \geqslant N_0} \langle n\rangle^{4\delta_*} w(2n)^2 \sup_{|z| \leqslant \pi/4} \left|\beta^{\pm}(n,z)\right|^2 \leqslant C;$$

(ii) $$\sum_{|n| \geqslant N_0} \langle n\rangle^{6\delta_*} w(2n)^2 \sup_{|z| \leqslant A/\langle n\rangle^{\delta_*}} \left|\beta^{\pm}(n,z) - \beta_0^{\pm}(n)\right|^2 \leqslant C.$$

Proof. Notice that $\beta^{\pm}(n,z) = \beta_0^{\pm}(n) + z\beta_{\#}^{\pm}(n,z) + \tilde{\beta}^{\pm}(n,z)$ and hence (i) is a consequence of Corollary 2.9 and formula (2.36). Statement (ii) is proved in the same fashion. As the supremum of $|\beta^{\pm}(n,z) - \beta_0^{\pm}(n)|$ is only taken over $|z| \leqslant A/\langle n\rangle^{\delta_*}$, the asymptotics of $z\beta_{\#}^{\pm}(n,z)$ can be improved by δ_* to obtain from formula (2.36)

$$\sum_{|n| \geqslant N_0} \langle n\rangle^{6\delta_*} w(2n)^2 \sup_{|z| \leqslant A/\langle n\rangle^{\delta_*}} \left|z\beta_{\#}^{\pm}(n,z)\right|^2 \leqslant C. \quad \square$$

2.6. *z-equation*

In view of (2.21), (2.22), and (2.29), the Q-equation leads to the following 2×2 system

$$\begin{pmatrix} -z + \alpha(n,z) & \hat{\psi}_2(2n) + \beta^{+}(n,z) \\ \hat{\psi}_1(-2n) + \beta^{-}(n,z) & -z + \alpha(n,z) \end{pmatrix} \begin{pmatrix} x^F \\ y^F \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.37}$$

Given $|n| \geqslant N$ and $|z| \leqslant \pi/2$, the number $\lambda = n\pi + z$ is a periodic eigenvalue of L iff there exists a nontrivial solution of (2.37) $(x^F, y^F) \in \mathbb{C}^2 \backslash (0,0)$, or, equivalently, iff the determinant of the 2×2 matrix in (2.37) vanishes,

$$\left(z - \alpha(n,z)\right)^2 - \left(\hat{\psi}_2(2n) + \beta^{+}(n,z)\right)\left(\hat{\psi}_1(-2n) + \beta^{-}(n,z)\right) = 0. \tag{2.38}$$

Proceeding similarly as in [7], Eq. (2.38) is solved in two steps: for ζ with $|\zeta| \leqslant \pi/8$ given, consider

$$z_n = \alpha(n, z_n) + \zeta. \tag{2.39}$$

Substituting a solution $z(\zeta) \equiv z_n(\zeta)$ of (2.39) into (2.38) leads to an equation for $\zeta \equiv \zeta_n$,

$$\zeta^2 - \left(\hat{\psi}_2(2n) + \beta^{+}\left(n, z(\zeta)\right)\right)\left(\hat{\psi}_1(-2n) + \beta^{-}\left(n, z(\zeta)\right)\right) = 0. \tag{2.40}$$

Eq. (2.39) is referred to as the z-equation and Eq. (2.40) as the ζ-equation.

In this section we deal with the z-equation (2.39). To solve it we use the contractive mapping principle. According to Lemma 2.7 we can choose $N_1 \geqslant N_0$ (with N_0 given by (2.18)) so that for any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$ and $|n| \geqslant N_1$

$$\sup_{|z| \leqslant \pi/2} \left|\alpha(n,z)\right| < \pi/8. \tag{2.41}$$

The following result can be proved by the same line of arguments used in the proof of [7, Proposition 1.6].

Proposition 2.11. *Let $M \geqslant 1$, $0 < \delta \leqslant 1/2$ and w be a δ-weight. Then, there exists $N_1 \geqslant N_0$ so that for any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$, $|\zeta| \leqslant \pi/8$ and $|n| \geqslant N_1$, Eq.* (2.39) *has a unique solution $z_n = z_n(\zeta)$ satisfying $|z_n| < \pi/4$. The solution depends analytically on ζ.*

2.7. *ζ-equation*

In this section, we prove the existence of solutions of the ζ-equation (2.40)

$$\zeta^2 - \big(\hat{\psi}_2(2n) + \beta^+(n, z(\zeta)\big)\big(\hat{\psi}_1(-2n) + \beta^-\big(n, z(\zeta)\big)\big) = 0$$

using Rouché's theorem. Introduce

$$r_n := \Big(\big|\hat{\psi}_2(2n)\big| + \sup_{|z|\leqslant\pi/2}\big|\beta^+(n,z)\big|\Big) \vee \Big(\big|\hat{\psi}_1(-2n)\big| + \sup_{|z|\leqslant\pi/2}\big|\beta^-(n,z)\big|\Big). \tag{2.42}$$

Using the same line of arguments used in the proof of [7, Proposition 1.15] one obtains the following

Proposition 2.12. *Let $M \geqslant 1$, $0 < \delta \leqslant 1/2$ and w be a δ-weight. Then there exists $N_2 \geqslant N_1$ so that, for any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$ and $|n| \geqslant N_2$, Eq.* (2.40) *has exactly two* (*counted with multiplicity*) *solutions ζ_n^+, ζ_n^- in $\overline{\mathcal{D}}_{r_n}$.*

2.8. *Proof of Theorem 1.1*

In this section Theorem 1.1 is proved.

Proof of Theorem 1.1(i). Let $z_n^\pm := z(\zeta_n^\pm) = \zeta_n^\pm + \alpha(n, z_n^\pm)$, where $\zeta_n^\pm$ are the two solutions of the ζ-equation provided by Proposition 2.12 ($|n| \geqslant N_2$). Then, for $|n| \geqslant N_2$

$$|z_n^+ - z_n^-| \leqslant |\zeta_n^+ - \zeta_n^-| + \sup_{|z|\leqslant\pi/4}\left|\frac{\mathrm{d}}{\mathrm{d}z}\alpha(n,z)\right||z_n^+ - z_n^-|. \tag{2.43}$$

As $N_2 \geqslant N_1$ and $|n| \geqslant N_2$ one has by the analyticity of $z \mapsto \alpha(n, z)$ and (2.41)

$$\sup_{|z|\leqslant\pi/4}\left|\frac{\mathrm{d}}{\mathrm{d}z}\alpha(n,z)\right| \leqslant \frac{1}{2}.$$

Together with $|\zeta_n^+ - \zeta_n^-| \leqslant |\zeta_n^+| + |\zeta_n^-| \leqslant 2r_n$ Eq. (2.43) then leads to

$$|z_n^+ - z_n^-| \leqslant 4r_n.$$

By the definition (2.42) of r_n, the estimates of $\beta_n^\pm$ in Proposition 2.10(i) and the identity $\lambda_n^+ - \lambda_n^- = z_n^+ - z_n^-$, the latter equation implies that there exists $C \geqslant 1$ such that, for any 1-periodic functions $\psi_1, \psi_2 \in H^w$, $\|\psi_j\|_w \leqslant M$,

$$\sum_{|n|\geqslant N_2} w(2n)^2|\lambda_n^+ - \lambda_n^-|^2 \leqslant C. \quad \square$$

Towards the proof of Theorem 1.1(ii), rewrite Eq. (2.40),

$$(\zeta_n^\pm)^2 - \rho_n^2 = \eta\big(n, z(\zeta_n^\pm)\big), \tag{2.44}$$

where

$$\rho_n = \big(\big(\hat\psi_2(2n) + \beta_0^+(n)\big)\big(\hat\psi_1(-2n) + \beta_0^-(n)\big)\big)^{1/2}$$

with an arbitrary but fixed choice of the square root and

$$\begin{aligned}\eta(n,z) = {}& \hat\psi_2(2n)\big(\beta^-(n,z) - \beta_0^-(n)\big) + \hat\psi_1(-2n)\big(\beta^+(n,z) - \beta_0^+(n)\big)\\ &+ \big(\beta^-(n,z) - \beta_0^-(n)\big)\big(\beta^+(n,z) - \beta_0^+(n)\big).\end{aligned} \tag{2.45}$$

In view of the definition (2.42) and as w is assumed to be a δ-weight, we have for some constant $C_1 \geqslant 1$ depending on δ and M

$$r_n \leqslant \frac{C_1}{\langle n\rangle^{\delta_*}} \quad \big(\forall |n| \geqslant N_0\big). \tag{2.46}$$

By Lemma 2.7 there exists $C_2 \geqslant 1$ depending on δ and M such that for $|n| \geqslant N_0$ and $|z| \leqslant \pi/2$

$$\big|\alpha(n,z)\big| \leqslant \frac{C_2}{\langle n\rangle^{\delta_*}}. \tag{2.47}$$

Let $A = C_1 + C_2$ and define

$$s_n := \sup_{|z| \leqslant 2A/\langle n\rangle^{\delta_*}} \big|\eta(n,z)\big|. \tag{2.48}$$

Notice that by Proposition 2.10(ii), there exists $C > 0$ so that

$$\sum_{|n| \geqslant N_2} \langle n\rangle^{3\delta_*} w(2n)^2 s_n \leqslant C. \tag{2.49}$$

Choose $N_3 \geqslant N_2$, depending on δ and M, so that

$$\frac{\langle n\rangle^{\delta_*}}{A}\sqrt{s_n} < \frac{1}{2}, \quad \forall |n| \geqslant N_3. \tag{2.50}$$

Lemma 2.13. *Let $M \geqslant 1$, $0 < \delta$ and w be a δ-weight. For 1-periodic functions ψ_1, ψ_2 in H^w with $\|\psi_j\|_w \leqslant M$ $(j = 1,2)$ and $|n| \geqslant N_3$,*

$$|\zeta_n^+ - \rho_n| + |\zeta_n^- + \rho_n| \leqslant 6\sqrt{s_n} \quad \textit{or} \quad |\zeta_n^+ + \rho_n| + |\zeta_n^- - \rho_n| \leqslant 6\sqrt{s_n}.$$

Proof. Without loss of generality assume that $\delta \leqslant 1/2$ and hence $\delta = \delta_*$. By (2.44) we have for $|n| \geqslant N_3$

$$(\zeta_n^\pm - \rho_n)(\zeta_n^\pm + \rho_n) = \eta\big(n, z(\zeta_n^\pm)\big). \tag{2.51}$$

By definition, $z(\zeta_n^\pm) = \zeta_n^\pm + \alpha(n, z(\zeta_n^\pm))$ and therefore from (2.46), (2.47) and Proposition 2.12, we conclude for $|n| \geqslant N_3$,

$$\big|z(\zeta_n^\pm)\big| \leqslant \frac{A}{\langle n\rangle^\delta}. \tag{2.52}$$

From the definition of s_n (see (2.48)) and (2.51) we deduce

$$|\zeta_n^\pm - \rho_n||\zeta_n^\pm + \rho_n| \leqslant s_n. \tag{2.53}$$

Thus $\min_\pm |\zeta_n^+ \pm \rho_n| \leqslant \sqrt{s_n}$ and $\min_\pm |\zeta_n^- \pm \rho_n| \leqslant s_n^{1/2}$. We distinguish two cases:

Case 1. $|\rho_n| \leqslant 2\sqrt{s_n}$. In this case $|\zeta_n^\pm - \rho_n| \leqslant \sqrt{s_n}$ implies

$$|\zeta_n^\pm + \rho_n| \leqslant |\zeta_n^\pm - \rho_n| + 2|\rho_n| \leqslant 5\sqrt{s_n}$$

and, similarly, $|\zeta_n^\pm + \rho_n| \leqslant \sqrt{s_n}$ implies $|\zeta_n^\pm - \rho_n| \leqslant 5\sqrt{s_n}$, thus Lemma 2.13 is proved in Case 1.

Case 2. $|\rho_n| > 2\sqrt{s_n}$. It suffices to show that it is impossible to have $\max_\pm |\zeta_n^\pm - \rho_n| \leqslant \sqrt{s_n}$, or $\max_\pm |\zeta_n^\pm + \rho_n| \leqslant \sqrt{s_n}$. To the contrary, assume that

$$\max_\pm |\zeta_n^\pm - \rho_n| \leqslant \sqrt{s_n}. \tag{2.54}$$

(The other case is treated in the same way.) By (2.54), $|\zeta_n^\pm + \rho_n| \geqslant 2|\rho_n| - \sqrt{s_n} > \frac{3}{2}|\rho_n|$, hence

$$|\zeta_n^+ + \zeta_n^-| \geqslant |\zeta_n^+ + \rho_n| - |\zeta_n^- - \rho_n| > |\rho_n|. \tag{2.55}$$

Divide

$$(\zeta_n^+)^2 - (\zeta_n^-)^2 = \eta\big(n, z(\zeta_n^+)\big) - \eta\big(n, z(\zeta_n^-)\big)$$

by $\zeta_n^+ + \zeta_n^-$ and use (2.55) and (2.52) to deduce

$$|\zeta_n^+ - \zeta_n^-| \leqslant \frac{1}{|\rho_n|} \sup_{|z| \leqslant A/\langle n\rangle^\delta} \left|\frac{\mathrm{d}\eta}{\mathrm{d}z}(n, z)\right| \big|z(\zeta_n^+) - z(\zeta_n^-)\big|. \tag{2.56}$$

To arrive at a contradiction we first show that $\zeta_n^+ - \zeta_n^- = 0$. As

$$\big|z(\zeta_n^+) - z(\zeta_n^-)\big| \leqslant |\zeta_n^+ - \zeta_n^-| + \sup_{|z| \leqslant \pi/2} \left|\frac{\mathrm{d}}{\mathrm{d}z}\alpha(n, z)\right| \big|z(\zeta_n^+) - z(\zeta_n^-)\big|,$$

(2.41) leads to ($|n| \geqslant N_2$)

$$\left|z(\zeta_n^+) - z(\zeta_n^-)\right| \leqslant 2|\zeta_n^+ - \zeta_n^-|. \tag{2.57}$$

On the other hand, as $z \mapsto \eta(n,z)$ is analytic in $\{z, |z| < \pi/2\}$, we have by Cauchy's inequality,

$$\sup_{|z| \leqslant A/\langle n\rangle^\delta} \left|\frac{\mathrm{d}}{\mathrm{d}z}\eta(n,z)\right| \leqslant \frac{\langle n\rangle^\delta}{A} \sup_{|z| \leqslant 2A/\langle n\rangle^\delta} \left|\eta(n,z)\right| \leqslant \frac{\langle n\rangle^\delta}{A} s_n. \tag{2.58}$$

Combining (2.56), (2.57) with (2.50) we obtain,

$$|\zeta_n^+ - \zeta_n^-| \leqslant \frac{2}{|\rho_n|}\frac{\langle n\rangle^\delta}{A} s_n|\zeta_n^+ - \zeta_n^-| \leqslant \frac{\langle n\rangle^\delta}{A}\sqrt{s_n}|\zeta_n^+ - \zeta_n^-| \leqslant \frac{1}{2}|\zeta_n^+ - \zeta_n^-|$$

and we conclude that $\zeta_n^+ = \zeta_n^- \equiv \zeta_n$. This contradicts the assumption $|\rho_n| > 2\sqrt{s_n}$ as one can see in the following way: by Eq. (2.44),

$$2\zeta_n = \frac{\mathrm{d}}{\mathrm{d}\zeta}\eta\big(n, z(\zeta_n)\big) = \frac{\mathrm{d}}{\mathrm{d}z}\eta\big(n, z(\zeta_n)\big) \cdot \frac{\mathrm{d}}{\mathrm{d}\zeta} z(\zeta_n).$$

By (2.58),

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\eta\big(n, z(\zeta_n)\big)\right| \leqslant \frac{\langle n\rangle^\delta}{A} s_n$$

and by (2.41),

$$\left|\frac{\mathrm{d}}{\mathrm{d}\zeta} z(\zeta)\right| = \left|\frac{\mathrm{d}}{\mathrm{d}\zeta}\big(\zeta + \alpha\big(n, z(\zeta)\big)\big)\right| \leqslant 1 + \frac{1}{2} \leqslant 2,$$

hence

$$|\zeta_n| \leqslant \frac{\langle n\rangle^\delta}{A} s_n \tag{2.59}$$

and, by (2.55),

$$|\rho_n| < 2|\zeta_n| \leqslant 2\frac{\langle n\rangle^\delta}{A} s_n < \sqrt{s_n},$$

where for the last inequality we used (2.50). □

Proof of Theorem 1.1(ii). Let N_3 be given by (2.50). Recall that

$$\lambda_n^+ - \lambda_n^- = z_n^+ - z_n^- = \zeta_n^+ - \zeta_n^- + \alpha\big(n, z(\zeta_n^+)\big) - \alpha\big(n, z(\zeta_n^-)\big).$$

By Lemma 2.13, for $|n| \geqslant N_3$,

$$\min_{\pm} \left|(\zeta_n^+ - \zeta_n^-) \pm 2\rho_n\right| \leqslant 6\sqrt{s_n}.$$

By the analyticity of $\alpha(n,z)$ and Lemma 2.7, for $|n| \geqslant N_0$,

$$\sup_{|z|\leqslant\pi/4} \left|\frac{\mathrm{d}}{\mathrm{d}z}\alpha(n,z)\right| \leqslant \frac{C}{\langle n\rangle^{2\delta}}.$$

Combining these two estimates, we get for $|n| \geqslant N_3$,

$$\begin{aligned}\min_{\pm} \left|(\lambda_n^+ - \lambda_n^-) \pm 2\rho_n\right| &\leqslant \min_{\pm} \left|(\zeta_n^+ - \zeta_n^-) \pm 2\rho_n\right| + \left(\sup_{|z|\leqslant\pi/2} \left|\frac{\mathrm{d}}{\mathrm{d}z}\alpha(n,z)\right|\right) \left|\lambda_n^+ - \lambda_n^-\right| \\ &\leqslant 6\sqrt{s_n} + C\frac{|\lambda_n^+ - \lambda_n^-|}{\langle n\rangle^{2\delta}}.\end{aligned}$$

Hence, by (2.49) and Theorem 1.1(i),

$$\sum_{|n|\geqslant N_3} \langle n\rangle^{3\delta} w(2n)^2 \min_{\pm} \left|(\lambda_n^+ - \lambda_n^-) \pm 2\rho_n\right|^2 \leqslant C. \qquad \square$$

2.9. Improvement of Theorem 1.1 for L selfadjoint

For ψ a 1-periodic functions in H^w, the operator $L(\psi,\overline{\psi})$ is selfadjoint. In this section we show that in this case the decay rate of the asymptotics in Theorem 1.1(ii) can be improved as follows:

Theorem 2.14. *Let $M \geqslant 1$, $\delta > 0$ and w be a δ-weight. Then there exist constants $1 \leqslant C < \infty$ and $1 \leqslant N < \infty$ so that for any $|n| \geqslant N$ and any 1-periodic function $\psi \in H^w$ with $\|\psi\|_w \leqslant M$,*

$$\sum_{|n|\geqslant N} \langle n\rangle^{4\delta_*} w(2n)^2 \min_{\pm} \left|(\lambda_n^+ - \lambda_n^-) \pm 2\rho_n\right|^2 \leqslant C.$$

Proof. Using the definition (2.13) with $\psi_1 = \psi$ and $\psi_2 = \overline{\psi}$ we get $\overline{R_n^{(1)}(k,j)(\overline{z})} = R_n^{(2)}(k,j)(z)$ and thus $\overline{\beta^-(n,z)} = \beta^+(n,\overline{z})$. As the eigenvalues $\lambda_n^\pm = n\pi + z(\zeta_n^\pm)$ of $L(\psi,\overline{\psi})$ are real, Eq. (2.40) then reads (with $|n| \geqslant N_2$ and N_2 as in Proposition 2.12)

$$\left(\zeta_n^\pm\right)^2 = \left|\hat{\psi}(2n) + \beta^+\left(n, z(\zeta_n^\pm)\right)\right|^2 = \left|\hat{\psi}(2n) + \beta_0^+(n) + \tilde{\beta}^+\left(n, z(\zeta_n^\pm)\right)\right|^2 \tag{2.60}$$

and ρ_n is given by (with an appropriate choice of the square root)

$$\rho_n = \left|\hat{\psi}(2n) + \beta_0^+(n)\right|.$$

Let $t_n := \sup_{|z|\leqslant A/\langle n\rangle^{\delta_*}} |\tilde{\beta}^+(n,z)|$, where $A := C_1 + C_2$ and C_1 are C_2 are defined by (2.46), (2.47). By Proposition 2.10(ii),

$$\sum_{|n|\geqslant N_0} \langle n\rangle^{6\delta_*} w(2n)^2 t_n^2 \leqslant C. \tag{2.61}$$

From (2.60) we deduce $\min_\pm |\zeta_n^+ \pm \rho_n| \leqslant t_n$ and $\min_\pm |\zeta_n^- \pm \rho_n| \leqslant t_n$. Substituting Lemma 2.15 below for Lemma 2.13, Theorem 2.14 follows in the same way as Theorem 1.1(ii). □

Define $N_4 \geqslant N_2$ such that

$$12\langle n\rangle^\delta t_n < A \quad \forall |n| \geqslant N_4.$$

Lemma 2.15. *Let $M \geqslant 1$, $\delta > 0$ and w be a δ-weight. For any 1-periodic function ψ in H^w with $\|\psi\|_w \leqslant M$ and $|n| \geqslant N_4$,*

$$|\zeta_n^+ - \rho_n| + |\zeta_n^- + \rho_n| \leqslant 6t_n \quad \textit{or} \quad |\zeta_n^+ + \rho_n| + |\zeta_n^- - \rho_n| \leqslant 6t_n.$$

Proof. The proof is similar to the one of Lemma 2.13. □

3. Riesz spaces and normal form of L

3.1. Riesz spaces

Let $M \geqslant 1$, $\delta > 0$ and w be a δ-weight. By Theorem 1.1, there exists $1 \leqslant N < \infty$ so that for any 1-periodic functions ψ_1, ψ_2 in H^w with $\|\psi_j\|_w \leqslant M$, the operator $L = L(\psi_1, \psi_2)$ has two (counted with multiplicity) periodic eigenvalues λ_n^+, λ_n^- near $n\pi$.

In Appendix A we introduce the periodic and antiperiodic boundary conditions bc Per^+ and bc Per^-. We point out that

$$\mathit{spec}\, L = \mathit{spec}\, L_{\mathrm{Per}^+} \cup \mathit{spec}\, L_{\mathrm{Per}^-}$$

and introduce the Riesz projectors $\Pi_{2n} : L^2([0,1];\mathbb{C}^2) \to L^2([0,1];\mathbb{C}^2)$, corresponding to bc Per^+ and $\Pi_{2n-1} : L^2([0,1];\mathbb{C}^2) \to L^2([0,1];\mathbb{C}^2)$, corresponding to bc Per^- ($n \in \mathbb{Z}$). Further denote by E_n the $\mathbb{C}$-vector spaces

$$E_n := \Pi_n\big(L^2([0,1];\mathbb{C}^2)\big) \quad \big(|n| \geqslant N\big).$$

Notice that $\dim_{\mathbb{C}} E_n = \operatorname{tr} \Pi_n = 2\ \forall |n| \geqslant N$. If $\lambda_n^+ \neq \lambda_n^-$ or $\lambda_n^+ = \lambda_n^-$ is of geometric multiplicity two, there exists a basis of E_n consisting of eigenfunctions F^+ and F^- corresponding to the eigenvalues $\lambda_n^\pm$. If $\lambda_n^+ = \lambda_n^-$ is of geometric multiplicity 1, E_n is the root space of λ_n^+. Denote by F a L^2-normalized eigenfunction of L corresponding to the eigenvalue $\lambda = n\pi + z$,

$$(L - \lambda)F = 0, \quad \|F\| = 1,$$

where $\|\cdot\|$ denotes the L^2-norm in $L^2([0,1];\mathbb{C}^2)$. Then

$$F(x) = x^F e_n^+(x) + y^F e_n^-(x) + \sum_{k\neq n} \bigl(\breve{F}_2(k)e_k^+(x) + \breve{F}_1(-k)e_k^-(x)\bigr),$$

where

$$\begin{pmatrix} \breve{F}_2 \\ J\breve{F}_1 \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} y^F \\ x^F \end{pmatrix}$$

with

$$\begin{aligned}
V_{11} &= (z - D_n)^{-1}(Id - P_n)^{-1} S^n \hat{\psi}_2,\\
V_{12} &= (z - D_n)^{-1} R_n^{(2)} (Id - Q_n)^{-1} S^n J\hat{\psi}_1,\\
V_{21} &= (z - D_n)^{-1} R_n^{(1)} (Id - P_n)^{-1} S^n \hat{\psi}_2,\\
V_{22} &= (z - D_n)^{-1}(Id - Q_n)^{-1} S^n J\hat{\psi}_1.
\end{aligned}$$

Proposition 3.1. *Let* $0 < \delta \leqslant 1$, $M \geqslant 1$ *and* w *be a* δ*-weight. Then there exist* $C \equiv C(\delta, M) \geqslant 1$ *and* $N \equiv N(M,\delta) \geqslant 1$ *such that for* 1*-periodic functions* $\psi_1, \psi_2 \in H^w$ *with* $\|\psi_j\|_w \leqslant M$ *and* $|n| \geqslant N$

(i) $1/2 \leqslant |x^F|^2 + |y^F|^2 \leqslant 1$;
(ii) $\|\breve{F}_2\| \leqslant 2\frac{C}{\langle n\rangle^\delta}$, $\|J\breve{F}_1\| \leqslant 2\frac{C}{\langle n\rangle^\delta}$, *where* $\|\cdot\|$ *stands for the* ℓ^2*-norm.*

Proof. As $\|F\| = 1$, we have

$$\|F\|^2 = |x^F|^2 + |y^F|^2 + \|\breve{F}_2\|^2 + \|J\breve{F}_1\|^2 = 1.$$

Hence

$$|x^F|^2 + |y^F|^2 \leqslant 1.$$

Further, by Proposition 2.4, for $|n| \geqslant N_0$

$$\|(Id - P_n)^{-1}\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant 2, \qquad \|(Id - Q_n)^{-1}\|_{\mathcal{L}(\ell^2_{S^n w})} \leqslant 2.$$

By Corollary 2.3, there exists $C > 1$ such that

$$\|R_n^{(j)}\|_{\mathcal{L}(\ell^2_{S^n w}, \ell^2)} \leqslant \frac{C}{\langle n\rangle^{\delta\wedge 1}}$$

and by the definition of D_n, for some $1 \leqslant C < \infty$,

$$\|(z - D_n)^{-1}\|_{\mathcal{L}(\ell^2_{S^n w}, \ell^2)} \leqslant \frac{C}{\langle n\rangle^{1\wedge\delta}}, \qquad \|(z - D_n)^{-1}\|_{\mathcal{L}(\ell^2, \ell^2)} \leqslant 1.$$

Hence for $|n| \geqslant N_0$

$$\|V_{11}\| + \|V_{22}\| \leqslant \frac{C}{\langle n\rangle^{\delta\wedge 1}}, \qquad \|V_{12}\| + \|V_{21}\| \leqslant \frac{C}{\langle n\rangle^{\delta\wedge 1}}$$

for some $1 < C < \infty$ and one concludes that

$$\|\breve{F}_2\| \leqslant \frac{C}{\langle n\rangle^{\delta\wedge 1}}\left(|x^F| + |y^F|\right) \leqslant 2\frac{C}{\langle n\rangle^{\delta\wedge 1}}, \qquad \|J\breve{F}_1\| \leqslant \frac{C}{\langle n\rangle^{\delta\wedge 1}}\left(|x^F| + |y^F|\right) \leqslant 2\frac{C}{\langle n\rangle^{\delta\wedge 1}}.$$

By choosing $N \geqslant N_0$ sufficiently large we have, for $|n| \geqslant N$,

$$\|\breve{F}_2\|^2 + \|J\breve{F}_1\|^2 \leqslant \frac{1}{2}$$

and hence $1/2 \leqslant |x^F|^2 + |y^F|^2$. $\square$

3.2. Normal form of L

In this section we want to derive a normal form of the restriction of L to the Riesz spaces E_n. For this purpose introduce an orthonormal basis of E_n as follows: choose $F \equiv F^+$ to be an L_2-normalized eigenfunction of L corresponding to the eigenvalue $\lambda^+ \equiv \lambda_n^+$ and $\Phi \in E_n$ with

$$(\Phi, F) = 0, \quad \|\Phi\| = 1,$$

where, as usual, $(\Phi, F) = \int_0^1 \overline{\Phi(x)} F(x)\,\mathrm{d}x$. In case λ^+ is a double eigenvalue,

$$\begin{pmatrix} L\Phi \\ LF \end{pmatrix} = \begin{pmatrix} \lambda^+ & \xi \\ 0 & \lambda^+ \end{pmatrix} \begin{pmatrix} \Phi \\ F \end{pmatrix}, \tag{3.1}$$

where $\xi \equiv \xi_n$ vanishes iff λ^+ is of geometric multiplicity two.

In case $\lambda_n^- \neq \lambda_n^+$, choose an L_2-normalized eigenfunction F^- of $\lambda^- \equiv \lambda_n^-$. Then

$$F^- = aF + b\Phi, \quad |a|^2 + |b|^2 = 1,\ b \neq 0.$$

With $\Phi = \frac{1}{b}F^- - \frac{a}{b}F$,

$$L\Phi = \lambda^- \frac{1}{b}F^- + \lambda^+ \frac{a}{b}F = \lambda^-\left(\frac{1}{b}F^- - \frac{a}{b}F\right) - \gamma\frac{a}{b}F,$$

where $\gamma \equiv \gamma_n := \lambda^+ - \lambda^-$. Hence

$$\begin{pmatrix} L\Phi \\ LF \end{pmatrix} = \begin{pmatrix} \lambda^- & \xi \\ 0 & \lambda^+ \end{pmatrix} \begin{pmatrix} \Phi \\ F \end{pmatrix} \tag{3.2}$$

with $\xi \equiv \xi_n := -\gamma\frac{a}{b}$. Notice that (3.1) and (3.2) have the same form. We refer to this form as the normal form of the restriction of L to the Riesz space E_n.

In the remaining part of this section we want to estimate the size of $(\xi_n)_{|n|\geqslant N}$. To this end, we write the equation $(L-\lambda^-)\Phi = \xi F$ in the basis e_k^+, e_k^- $(k \in \mathbb{Z})$. With

$$\Phi = x^\Phi e_n^+ + y^\Phi e_n^- + \sum_{k\neq n} \breve{\Phi}_2(k) e_k^+ + \breve{\Phi}_1(-k) e_k^- \quad \text{and}$$

$$F = x^F e_n^+ + y^F e_n^- + \sum_{k\neq n} \breve{F}_2(k) e_k^+ + \breve{F}_1(-k) e_k^-,$$

we then obtain the following inhomogeneous system (cf. (2.8)–(2.10))

$$-z^- x^\Phi + \hat{\psi}_2(2n) y^\Phi + \langle S^n \hat{\psi}_2, J\breve{\Phi}_1\rangle = \xi x^F, \tag{3.3}$$

$$\hat{\psi}_1(-2n) x^\Phi - z^- y^\Phi + \langle S^n J\hat{\psi}_1, \breve{\Phi}_2\rangle = \xi y^F, \tag{3.4}$$

$$\begin{pmatrix} y^\Phi (S^n\hat{\psi}_2)_{\mathbb{Z}\setminus n} \\ x^\Phi (S^n J\hat{\psi}_1)_{\mathbb{Z}\setminus n}\end{pmatrix} + (A_n - z^-)\begin{pmatrix}\breve{\Phi}_2 \\ J\breve{\Phi}_1\end{pmatrix} = \xi \begin{pmatrix}\breve{F}_2 \\ J\breve{F}_1\end{pmatrix}, \tag{3.5}$$

where, as usual, $\lambda_n^- \equiv \lambda^- = n\pi + z^-$. We use the above system to obtain an estimate for $\xi \equiv \xi_n$.

Write $\breve{\Phi} = (\breve{\Phi}_2, J\breve{\Phi}_1)$ and $\breve{F} = (\breve{F}_2, J\breve{F}_1)$. Recall that w is assumed to be a δ-weight and hence by Corollary 2.5, Eq. (3.5) (with $|n| \geqslant N_0$) can be solved for $\breve{\Phi}$,

$$\breve{\Phi} = (z^- - A_n)^{-1}\begin{pmatrix} y^\Phi (S^n\hat{\psi}_2)_{\mathbb{Z}\setminus n} \\ x^\Phi (S^n J\hat{\psi}_1)_{\mathbb{Z}\setminus n}\end{pmatrix} - \xi (z^- - A_n)^{-1}\breve{F}.$$

In this form, $\breve{\Phi}$ is substituted into (3.3), (3.4) to obtain (cf. Corollary 2.5)

$$\begin{aligned}&\begin{pmatrix} -z^- + \alpha(n,z^-) & \hat{\psi}_2(2n) + \beta^+(n,z^-) \\ \hat{\psi}_1(-2n) + \beta^-(n,z^-) & -z^- + \alpha(n,z^-)\end{pmatrix}\begin{pmatrix} x^\Phi \\ y^\Phi\end{pmatrix} \\ &\quad = \xi\begin{pmatrix} x^F \\ y^F\end{pmatrix} + \xi\left\langle \begin{pmatrix} S^n\hat{\psi}_2 \\ S^n J\hat{\psi}_1\end{pmatrix}, (z^- - A_n)^{-1}\breve{F}\right\rangle.\end{aligned} \tag{3.6}$$

Denote the right side of (3.6) by *RS*. By Corollary 2.5 $(z^- - A_n)^{-1}$ is uniformly bounded for $|n|$ sufficiently large and by Proposition 3.1, for $|n| \geqslant N$,

$$\frac{1}{2} \leqslant |x^F|^2 + |y^F|^2, \qquad \|\breve{F}\| \leqslant \frac{C}{\langle n\rangle^\delta}.$$

Hence *RS* can be estimated from below: There exists $1 \leqslant C \equiv C_{\delta,M} < \infty$ so that for $|n| \geqslant N$ (N as in Proposition 3.1)

$$RS \geqslant |\xi|\left(\frac{1}{\sqrt{2}} - \frac{C}{\langle n\rangle^\delta}\right).$$

By choosing N larger if necessary, we can assume that

$$\frac{1}{\sqrt{2}} - \frac{C}{\langle n\rangle^\delta} \geqslant \frac{1}{2} \quad \forall |n| \geqslant N \tag{3.7}$$

and (3.6) leads to

$$|\xi_n| \leqslant 4\big(|\zeta_n^-| + |\hat{\psi}_1(-2n)| + |\hat{\psi}_2(2n)| + |\beta^+(n, z^-)| + |\beta^-(n, z^-)|\big), \tag{3.8}$$

where we used that $|x^\Phi|^2 + |y^\Phi|^2 \leqslant 1$ and $\zeta_n^- = z^- - \alpha(n, z^-)$ with $z^- \equiv z(\zeta_n^-)$.

In view of Proposition 2.10 and Lemma 2.13, one then concludes from (3.8) the following

Proposition 3.2. *Let $M \geqslant 1$, $0 < \delta$, and w be a δ-weight. Then there exist $1 \leqslant N < \infty$, $1 \leqslant C = C_\delta < \infty$ such that for any 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w \leqslant M$*

$$\sum_{|n|\geqslant N} w(2n)^2 |\xi_n|^2 \leqslant C.$$

4. Dirichlet eigenvalues

4.1. Dirichlet boundary value problem

Consider the Zakharov–Shabat operator $L \equiv L(\psi_1, \psi_2)$ on the interval $[0, 1]$.

Definition 4.1. $F = (F_1, F_2) \in H^1([0, 1]; \mathbb{C}^2)$ satisfies Dirichlet boundary conditions if

$$F_1(0) - F_2(0) = 0, \qquad F_1(1) - F_2(1) = 0. \tag{4.1}$$

We mention that the Dirichlet boundary conditions take a more familiar form when the operator L is written as an *AKNS* operator L_{AKNS}

$$L_{AKNS} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} -q & p \\ p & q \end{pmatrix}, \tag{4.2}$$

where (ψ_1, ψ_2) and (p, q) are related by

$$\psi_1 = -q + \mathrm{i}p, \qquad \psi_2 = -q - \mathrm{i}p.$$

If $F = (F_1, F_2) \in H^1([0, 1]; \mathbb{C}^2)$ satisfies $LF = \lambda F$, then $L\widetilde{F} = \lambda \widetilde{F}$ where $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2)$ is given by

$$\widetilde{F}_1 = \frac{1}{\sqrt{2}\mathrm{i}}(F_1 + F_2), \qquad \widetilde{F}_2 = \frac{1}{\sqrt{2}}(F_2 - F_1).$$

The Dirichlet boundary conditions (4.1) then take the familiar form

$$\widetilde{F}_2(0) = 0, \qquad \widetilde{F}_2(1) = 0.$$

For the remaining part of Section 4, let $M \geqslant 1$, $\delta > 0$, and a δ-weight w be given as well as arbitrary 1-periodic functions $\psi_1, \psi_2 \in H^w$ with $\|\psi_j\|_w < M$. In Appendix A we have introduced, for $|n| \geqslant N$ with

N given by Theorem 1.1, the Riesz projectors Π_{2n}, Π_{2n-1} corresponding to periodic resp. antiperiodic boundary value problem on [0, 1] for L and the two dimensional subspaces $E_n = \mathit{Range}(\Pi_n)$.

The following proposition assures that there exists a 1-dimensional subspace of E_n which satisfies Dirichlet boundary conditions. Let (F, Φ) denote the orthonormal basis of $E_n \subseteq L^2([0,1];\mathbb{C}^2)$, introduced in Section 3.2.

Proposition 4.2. *For any $|n| \geqslant N$, there exists $G = (G_1, G_2) \in E_n$*

$$G = \alpha F + \beta \Phi, \quad |\alpha|^2 + |\beta|^2 = 1$$

which satisfies Dirichlet boundary conditions

$$G_1(0) - G_2(0) = 0, \qquad G_1(1) - G_2(1) = 0.$$

Proof. First consider the case where F satisfies $F_1(0) - F_2(0) = 0$. As F is either periodic or antiperiodic we conclude that $F_1(1) - F_2(1) = 0$ as well and thus $G := F$ has the required properties. If $F_1(0) - F_2(0) \neq 0$, notice that

$$\widetilde{G}(x) := \big(F_1(0) - F_2(0)\big)\Phi(x) - \big(\Phi_1(0) - \Phi_2(0)\big)F(x)$$

satisfies Dirichlet boundary conditions. As $\widetilde{G} \not\equiv 0$, we may define $G := \widetilde{G}/\|\widetilde{G}\|$. $\square$

By (3.1), (3.2), $L\Phi = \lambda^- \Phi + \xi F$ and $LF = \lambda^+ F$, hence, with $\gamma \equiv \gamma_n = \lambda^+ - \lambda^-$ and $\lambda \equiv \lambda^+$

$$LG = \alpha\lambda F + \beta L\Phi = \lambda G - \beta\gamma\Phi + \beta\xi F. \tag{4.3}$$

For $|n| \geqslant N$ sufficiently large, $\xi \equiv \xi_n$ and $\gamma \equiv \gamma_n$ are small and G is almost a Dirichlet eigenfunction. In the next sections we prove that $\lambda \equiv \lambda_n^+$ and G are good approximations of the Dirichlet eigenvalue $\mu \equiv \mu_n$ respectively Dirichlet eigenfunction H.

4.2. Decomposition

Let L_{Dir} denote the closed operator $L_{\mathrm{Dir}} = L(\psi_1, \psi_2)$ with domain

$$\mathit{dom}\, L_{\mathrm{Dir}} := \big\{F \in H^1[0,1] \mid F_1(0) - F_2(0) = 0, F_1(1) - F_2(1) = 0\big\}.$$

Let us fix n with $|n| \geqslant N$ (N as in Theorem 1.1). $\Pi_{\mathrm{Dir}} \equiv \Pi_{n,\mathrm{Dir}}$ denotes the Riesz projector

$$\Pi_{\mathrm{Dir}} := \frac{1}{2\pi \mathrm{i}} \int_{|z-n\pi|=\pi/2} (z - L_{\mathrm{Dir}})^{-1}\, \mathrm{d}z$$

acting on $L^2([0,1];\mathbb{C}^2)$ (cf. Appendix A). Let $\Omega_{\mathrm{Dir}} := \mathit{Id} - \Pi_{\mathrm{Dir}}$.

Notice that

$$\mathit{Range}\, \Pi_{\mathrm{Dir}} = \{aH \mid a \in \mathbb{C}\},$$

where $H \in dom\, L_{\mathrm{Dir}}$ is an $L^2[0,1]$-normalized eigenfunction for the Dirichlet eigenvalue $\mu \equiv \mu_n$,

$$L_{\mathrm{Dir}} H = \mu H, \quad \|H\| = 1.$$

Let $\chi \in \mathbb{C}$ with $|\chi| \leqslant 1$ defined by $\Pi_{\mathrm{Dir}} G = \chi H$ where G is given by Proposition 4.2. We have

$$G = \chi H + \Omega_{\mathrm{Dir}} G.$$

As G and H are in $dom\, L_{\mathrm{Dir}}$, $\Omega_{\mathrm{Dir}} G \in dom\, L_{\mathrm{Dir}}$ and

$$L_{\mathrm{Dir}} G = \chi \mu H + L_{\mathrm{Dir}} \Omega_{\mathrm{Dir}} G = \chi \mu H + \Omega_{\mathrm{Dir}} L_{\mathrm{Dir}} \Omega_{\mathrm{Dir}} G. \tag{4.4}$$

Where for the last equality we have used, $\Pi_{\mathrm{Dir}} L_{\mathrm{Dir}} \Omega_{\mathrm{Dir}} G = 0$, as L_{Dir} and Π_{Dir} commute on $dom\, L_{\mathrm{Dir}}$ and $\Pi_{\mathrm{Dir}} \Omega_{\mathrm{Dir}} = 0$.

On the other hand by (4.3),

$$LG = \lambda G + R, \quad R = -\beta\gamma\Phi + \beta\xi F$$

and thus, with $G = \chi H + \Omega_{\mathrm{Dir}} G$,

$$L_{\mathrm{Dir}} G = \lambda\chi H + \lambda\Omega_{\mathrm{Dir}} G + (\Pi_{\mathrm{Dir}} + \Omega_{\mathrm{Dir}}) R. \tag{4.5}$$

Comparing the decompositions of the right sides of (4.4) and (4.5) leads to the following

Lemma 4.3.

$$\chi(\mu - \lambda) H = \Pi_{\mathrm{Dir}} R, \tag{4.6}$$

$$\big(L_{\mathrm{Dir}} - \lambda\big)\big(\Omega_{\mathrm{Dir}} G\big) = \Omega_{\mathrm{Dir}} R, \tag{4.7}$$

where R is given by

$$R = -\beta\gamma\Phi + \beta\xi F. \tag{4.8}$$

4.3. Proof of Theorem 1.2

Eqs (4.6)–(4.8) are now used to obtain estimates for $|\mu_n - \lambda_n^+|$ ($|n| \geqslant N$). For this we need to establish that $|\chi| \leqslant 1$ is bounded away from 0 and that $\|\Pi_{\mathrm{Dir}} R\|$ is small. The latter is easily seen as $\|R\| \leqslant |\gamma| + |\xi|$. To verify that $|\chi|$ is bounded away from 0 we show that $\Omega_{\mathrm{Dir}} G = G - \chi G$ is small. This is proved by using Eq. (4.7).

Lemma 4.4. *There exists $N \geqslant 1$ so that*

$$|\chi_n| \geqslant \frac{1}{2} \quad \forall |n| \geqslant N.$$

Proof. As $G = \chi H + \Omega_{\mathrm{Dir}} G$,

$$|\chi| \|H\| = \|G\| - \|\Omega_{\mathrm{Dir}} G\| = 1 - \|\Omega_{\mathrm{Dir}} G\|.$$

By Lemmas 4.3 and A.2 (for (4.9)), Proposition 3.2 (for (4.10)), and Theorem 1.1(i) (for (4.11)) there exist $1 \leqslant N < \infty$ and $1 \leqslant C < \infty$ so that for $|n| \geqslant N$

$$\|\Omega_{\mathrm{Dir}} G\| = \|(L_{\mathrm{Dir}} - \lambda)^{-1}(\Omega_{\mathrm{Dir}} R)\| \leqslant C\|R\| \leqslant C\big(|\xi_n| + |\gamma_n|\big), \tag{4.9}$$

$$|\xi_n| \leqslant \frac{C}{\langle n\rangle^{\delta}}, \tag{4.10}$$

$$|\gamma_n| \leqslant \frac{C}{\langle n\rangle^{\delta}} \tag{4.11}$$

(where for the last two inequalities we used that w is a δ-weight). Combining the above inequalities shows that for $|n|$ large enough

$$|\chi_n| \geqslant \frac{1}{2}. \quad \Box$$

Proof of Theorem 1.2. By (4.6),

$$|\chi||\mu - \lambda|\|H\| = \|\Pi_{\mathrm{Dir}} R\|.$$

By Lemma 4.4 there exists $N \geqslant 1$ so that for $|n| \geqslant N$

$$|\mu_n - \lambda_n^+| \leqslant 2C\big(|\xi_n| + |\gamma_n|\big),$$

where we have used that $\|\Pi_{\mathrm{Dir}}\| \leqslant C$ (cf. Lemma A.2). The claimed estimate then follows from the estimates of ξ_n (Proposition 3.2) and of γ_n (Theorem 1.1(i)). $\Box$

Appendix A: Spectral properties of $L(\psi_1, \psi_2)$

In this appendix we consider the operator $L(\psi_1, \psi_2)$ ($\psi_1,\ \psi_2$ 1-periodic functions in $L^2([0,2], \mathbb{C}^2)$) with various boundary conditions. For bc $\in \{\mathrm{Dir}, \mathrm{Per}^{\pm}, \mathrm{Per}\}$ denote by L_{bc} the Zakharov–Shabat operator $L_{\mathrm{bc}} = L(\psi_1, \psi_2)$ with the following domains:

$$\mathit{dom}\, L_{\mathrm{Dir}} := \big\{F \in H^1[0,1] \mid F_1(0) - F_2(0) = 0,\ F_1(1) - F_2(1) = 0\big\},$$

$$\mathit{dom}\, L_{\mathrm{Per}^+} := \big\{F \in H^1[0,1] \mid F(0) = F(1)\big\},$$

$$\mathit{dom}\, L_{\mathrm{Per}^-} := \big\{F \in H^1[0,1] \mid F(0) = -F(1)\big\}.$$

The operator $L \equiv L_{\mathrm{Per}}$ is defined on the interval $[0,2]$ and has the following domain,

$$\mathit{dom}\, L_{\mathrm{Per}} := \big\{F \in H^1[0,2] \mid F(0) = F(2)\big\}.$$

Let $spec_{\mathrm{bc}} \equiv spec(L_{\mathrm{bc}})$ be the spectrum of L_{bc}. For potentials $\psi_1 = \psi_2 \equiv 0$, i.e., $L_0 := L(0,0) = \mathrm{i}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\frac{\mathrm{d}}{\mathrm{d}x}$, $spec_{\mathrm{bc}}(L_0)$ can be given explicitely:

$$spec_{\mathrm{Dir}}(L_0) = \{k\pi \mid k \in \mathbb{Z}\}, \tag{A.1}$$

$$spec_{\mathrm{Per}^+}(L_0) = \{2k\pi \mid k \in \mathbb{Z}\}, \tag{A.2}$$

$$spec_{\mathrm{Per}^-}(L_0) = \big\{2(k+1)\pi \mid k \in \mathbb{Z}\big\}, \tag{A.3}$$

$$spec_{\mathrm{Per}}(L_0) = \{k\pi \mid k \in \mathbb{Z}\}. \tag{A.4}$$

Proposition A.1. *Let $\delta > 0$, $M \geqslant 1$ and w be a δ-weight. There exists an even integer N such that for any 1-periodic functions ψ_1 and ψ_2 in H^w, $\|\psi_j\|_w \leqslant M$, the following statements hold*:

(i) *for* $\mathrm{bc} \in \{\mathrm{Dir}, \mathrm{Per}^{\pm}, \mathrm{Per}\}$,

$$spec_{\mathrm{bc}} \subset \big\{\lambda \in \mathbb{C} \mid |\lambda| < N\pi - \pi/2\big\} \cup \Bigg(\bigcup_{|k| \geqslant N} \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| < \pi/2\}\Bigg);$$

(ii) *for $|k| \geqslant N$, $spec_{\mathrm{Per}} \cap \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| < \pi/2\}$ contains exactly one isolated pair of eigenvalues*;

(iii) *for $|k| \geqslant N$ and* $\mathrm{bc} := \mathrm{Per}^+$ (*k even*) *and* $\mathrm{bc} := \mathrm{Per}^-$ (*k odd*), *$spec_{\mathrm{bc}} \cap \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| < \pi/2\}$ contains exactly one isolated pair of eigenvalues*;

(iv) *for $|k| \geqslant N$, $spec_{\mathrm{Dir}} \cap \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| < \pi/2\}$ contains exactly one eigenvalue*;

(v) *the cardinality N_{bc} of $spec_{\mathrm{bc}} \cap \{\lambda \in \mathbb{C} \mid |\lambda| < N\pi - \pi/2\}$ is equal to $4N-2$ for* $\mathrm{bc} = \mathrm{Per}$, *$2N-1$ for* $\mathrm{bc} = \mathrm{Dir}$, *$2N-2$ for* $\mathrm{bc} = \mathrm{Per}^+$ *and $2N$ for* $\mathrm{bc} = \mathrm{Per}^-$.

As $spec_{\mathrm{Per}^+} \cup spec_{\mathrm{Per}^-} \subseteq spec_{\mathrm{Per}}$, Proposition A.1 implies

$$spec_{\mathrm{Per}} = spec_{\mathrm{Per}^+} \cup spec_{\mathrm{Per}^-}.$$

Proof. Define for $n \geqslant 1$, the union of contours,

$$\mathcal{R}_n = \big\{\lambda \in \mathbb{C} \mid |\lambda| = n\pi - \pi/2\big\} \cup \Bigg(\bigcup_{|k| > n} \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| = \pi/2\}\Bigg).$$

By (A.1)–(A.4), $(L_0 - \lambda) : dom(L_{\mathrm{bc}}) \to L^2$ is invertible for any $\lambda \in \mathcal{R}_n$, hence

$$(L - \lambda) = (L_0 - \lambda)\big(Id + Q_\lambda\big), \tag{A.5}$$

where

$$Q_\lambda = (L_0 - \lambda)^{-1} \begin{pmatrix} 0 & \psi_1 \\ \psi_2 & 0 \end{pmatrix}.$$

Using the orthogonal decomposition of L^2 by the eigenfunctions of $(L_0)_{\mathrm{bc}}$ and the assumption that w is a δ-weight, one gets (with $\mathcal{L} \equiv \mathcal{L}(L^2)$)

$$\|Q_\lambda\|_{\mathcal{L}} \leqslant M\left(\sum_{k\in\mathbb{Z}} \frac{1}{|k|^{2\delta}|k\pi-\lambda|^2}\right)^{1/2}. \tag{A.6}$$

As $\max_{k\in\mathbb{Z}}(1/(|k|^{2\delta}\langle k-n\rangle^{1/2}))^{1/2} \leqslant 1/\langle n\rangle^{\delta\wedge 1/4}$, one deduces from (A.6) that, for $\lambda \in \mathcal{R}_n$,

$$\|Q_\lambda\|_{\mathcal{L}} \leqslant \frac{M}{\langle n\rangle^{\delta\wedge 1/4}}\left(\sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^{3/2}}\right)^{1/2}. \tag{A.7}$$

Let N be an even integer such that

$$\frac{M}{\langle n\rangle^{\delta\wedge 1/4}}\left(\sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^{3/2}}\right)^{1/2} \leqslant \frac{1}{2}.$$

Then, for $\lambda \in \mathcal{R}_n$ with $n \geqslant N$

$$\|Q_\lambda\|_{\mathcal{L}} \leqslant \frac{1}{2}. \tag{A.8}$$

Combining (A.5) and (A.8), one deduces that $(L-\lambda)\colon dom(L_{\mathrm{bc}}) \to L^2$ is invertible for any $\lambda \in \mathcal{R}_n$ $(n \geqslant N)$ and any 1-periodic functions ψ_1, ψ_2 in H^w with $\|\psi_j\|_w \leqslant M$. In particular, $\mathcal{R}_n$ $(n \geqslant N)$ is contained in the resolvent set of $L_{\mathrm{bc}}(t\psi_1, t\psi_2)$ for any $0 \leqslant t \leqslant 1$. Hence the number of eigenvalues of $L_{\mathrm{bc}}(t\psi_1, t\psi_2)$ in each connected component of the interior of $\mathcal{R}_n$ stays the same for any $0 \leqslant t \leqslant 1$. To see that all eigenvalues are inside $\mathcal{R}_n$ one chooses n bigger and bigger. □

It follows from Proposition A.1 that the Riesz projectors Π_n and $\Pi_{n,\mathrm{Dir}}$ are well defined (for any $|n| \geqslant N$ and 1-periodic functions ψ_1, ψ_2 with $\|\psi_j\|_w \leqslant M$)

$$\Pi_n := \frac{1}{2\pi \mathrm{i}}\int_{|\lambda-n\pi|=\pi/2} (z-L_{\mathrm{Per}^+})^{-1}\,\mathrm{d}z \quad (n \text{ even}, \ |n| \geqslant N),$$
$$\Pi_n := \frac{1}{2\pi \mathrm{i}}\int_{|\lambda-n\pi|=\pi/2} (z-L_{\mathrm{Per}^-})^{-1}\,\mathrm{d}z \quad (n \text{ odd}, \ |n| \geqslant N)$$

and

$$\Pi_{n,\mathrm{Dir}} := \frac{1}{2\pi \mathrm{i}}\int_{|\lambda-n\pi|=\pi/2} (z-L_{\mathrm{Dir}})^{-1}\,\mathrm{d}z \quad (|n| \geqslant N),$$

where the contours $\{\lambda \mid |\lambda-n\pi| = \pi/2\}$ in the integrals above are counter-clockwise oriented. Furthermore, using (A.5) and (A.8), one deduces

Lemma A.2. *Assume that the assumptions of Proposition* A.1 *hold. Then there exists a constant* $1 \leqslant C \leqslant \infty$ *such that for any* $|n| \geqslant N$ (*with* N *as in Proposition* A.1)

$$\|\Pi_n\|_{\mathcal{L}(L^2[0,1])} \leqslant C \quad \textit{and} \quad \|\Pi_{n,\mathrm{Dir}}\|_{\mathcal{L}(L^2[0,1])} \leqslant C.$$

Appendix B: Proof of Lemma 2.8

Without loss of generality we may assume that $\delta_* = \delta$.

(i) As w_* is submultiplicative, one has

$$w(2n) = \langle 2n\rangle^{\delta} w_*(2n) \leqslant 2^{\delta}\langle n\rangle^{\delta} w_*(n+k)w_*(k+j)w_*(j+n) \tag{B.1}$$

and, by assumption, $\langle n\rangle^{\alpha} d_n(k) \leqslant d(k)$ $(\forall k)$. This leads to

$$\begin{aligned}\langle n\rangle^{2\delta+\alpha} w(2n)\Psi_n(a,b,d_n)(2n) &\leqslant \langle n\rangle^{2\delta} w(2n) \sum_k \frac{a(k+n)}{\langle k-n\rangle} \sum_j \frac{b(k+j)}{\langle j-n\rangle} d(j+n)\\ &\leqslant \sum_{k,j} K_n(k,j)\widetilde{a}(k+n)\widetilde{b}(k,j)\widetilde{d}(j+n),\end{aligned} \tag{B.2}$$

where for any $u \in \ell^2_w$ we denote by $\widetilde{u}$ the ℓ^2-sequence $\widetilde{u}(j) := w(j)u(j)$ and $K_n(k,j)$ is given by

$$K_n(k,j) := \frac{2^{\delta}\langle n\rangle^{3\delta}}{\langle k-n\rangle\langle j-n\rangle\langle k+n\rangle^{\delta}\langle k+j\rangle^{\delta}\langle j+n\rangle^{\delta}}.$$

Notice that $K_n(k,j)$ is symmetric in k and j. To estimate $K_n(k,j)$ we need to consider four different regions:

Estimate of $K_n(k,j)$ *in* $|k-n| < |n|/2,\ |j-n| < |n|/2$: In this case

$$|k+n| \geqslant |n|, \quad |j+n| \geqslant |n|, \quad |k+j| \geqslant 2|n| - |k-n| - |j-n| \geqslant |n|,$$

hence

$$K_n(k,j) \leqslant \frac{2^{\delta}}{\langle k-n\rangle\langle j-n\rangle} \leqslant \frac{1}{\langle k-n\rangle^2} + \frac{1}{\langle j-n\rangle^2}.$$

Estimate of $K_n(k,j)$ *in* $|k-n| \geqslant |n|/2,\ |j-n| < |n|/2$: In this case

$$|k-n| \geqslant \frac{|n|}{2}, \quad |j+n| > |n|,$$

hence

$$K_n(k,j) \leqslant \frac{2^{\delta}}{\langle j-n\rangle\langle k+j\rangle^{\delta}}.$$

Estimate of $K_n(k,j)$ in $|k-n| < |n|/2,\ |j-n| \geqslant |n|/2$: Using the symmetry of $K_n(k,j)$ in k and j, the latter estimate leads to

$$K_n(k,j) \leqslant \frac{2^\delta}{\langle k-n\rangle\langle k+j\rangle^\delta}.$$

Estimate of $K_n(k,j)$ in $|k-n| \geqslant |n|/2,\ |j-n| \geqslant |n|/2$: We get

$$K_n(k,j) \leqslant \frac{16^\delta}{\langle k-n\rangle^{1-\delta}\langle k+j\rangle^\delta\langle j+n\rangle^\delta}.$$

Combining the above estimates one obtains for $k,j,n \in \mathbb{Z}$,

$$K_n(k,j) \leqslant \frac{1}{\langle k-n\rangle^2} + \frac{1}{\langle j-n\rangle^2} + \frac{2}{\langle k+j\rangle^\delta}\frac{1}{\langle k-n\rangle} + \frac{4}{\langle k-n\rangle^{1-\delta}\langle k+j\rangle^\delta\langle j+n\rangle^\delta}.$$

Therefore

$$\begin{aligned}\sum_{k,j} K_n(k,j)\widetilde{a}(k+n)\widetilde{b}(k+j)\widetilde{d}(j+n) &\leqslant \left(\widetilde{a} * \frac{1}{\langle k\rangle^2}(J\widetilde{b} * \widetilde{d})\right)(2n) + \left(\widetilde{d} * \frac{1}{\langle k\rangle^2}(J\widetilde{b} * \widetilde{a})\right)(2n) \\ &\quad + 2\left(\widetilde{a} * \frac{1}{\langle k\rangle}\left(\frac{J\widetilde{b}}{\langle k\rangle^\delta} * \widetilde{d}\right)\right)(2n) + 4\left(\widetilde{a} * \frac{1}{\langle k\rangle^{1-\delta}}\left(\frac{J\widetilde{b}}{\langle k\rangle^\delta} * \frac{\widetilde{d}}{\langle k\rangle^\delta}\right)\right)(2n), \qquad \text{(B.3)}\end{aligned}$$

where for $u \in \ell^2(\mathbb{Z})$ and $\eta \geqslant 0$, $u/\langle k\rangle^\eta$ denotes the sequence given by $(u/\langle k\rangle^\eta)(j) := u(j)/\langle j\rangle^\eta$ $(\forall j)$. Using the standard convolution estimates $\|u * v\|_{\ell^2} \leqslant \|u\|_{\ell^1}\|v\|_{\ell^2}$ and $\|u * v\|_{\ell^\infty} \leqslant \|u\|_{\ell^2}\|v\|_{\ell^2}$ for the first two terms on the right side of (B.3), Corollary B.2(i) for the third term and Corollary B.2(ii) for the last term on the right side of (B.3), one obtains from (B.2)

$$\sum_n \left(\langle n\rangle^{2\delta+\alpha} w(2n)\Psi_n(a,b,d_n)(2n)\right)^2 \leqslant C\|a\|_w\|b\|_w\|d\|_w$$

for a constant $1 \leqslant C \leqslant C_\delta < \infty$ only depending on δ.

(ii) Using (B.1) and the assumption $\langle n\rangle^\alpha d_n(k) \leqslant d(k)$ $(\forall k)$ we get

$$\begin{aligned}\langle n\rangle^{\delta+\alpha} w(n+\ell)\Psi_n(a,b,d_n)(\ell+n) &\leqslant \langle n\rangle^\delta w(n+\ell)\sum_{k,j}\frac{a(k+\ell)}{\langle k-n\rangle}\frac{b(k+j)}{\langle k-j\rangle}d(j+n) \\ &\leqslant \sum_{k,j} H_n(\ell,k,j)\widetilde{a}(k+\ell)\widetilde{b}(k+j)\widetilde{d}(j+n),\end{aligned}$$

where $H_n(\ell,k,j)$ is given by

$$H_n(\ell,k,j) := \frac{\langle n\rangle^\delta\langle \ell+n\rangle^\delta}{\langle k-n\rangle\langle j-n\rangle\langle k+\ell\rangle^\delta\langle k+j\rangle^\delta\langle j+n\rangle^\delta}.$$

To estimate $H_n(\ell,k,j)$ we need to consider two different regions:

Estimate of $H_n(\ell,k,j)$ *in* $|j-n|<|n|/2$: In this case

$$|j+n|>|n|,\quad \langle \ell+n\rangle^{\delta}\leqslant\langle \ell+k\rangle^{\delta}\langle -k+n\rangle^{\delta},$$

hence

$$H_n(\ell,k,j)\leqslant\frac{1}{\langle k-n\rangle^{1-\delta}\langle j-n\rangle\langle k+j\rangle^{\delta}\langle j+n\rangle^{\delta}}\leqslant\frac{1}{\langle k-n\rangle^{1-\delta}\langle k+j\rangle^{\delta}\langle j+n\rangle^{\delta}}.$$

Estimate of $H_n(\ell,k,j)$ *in* $|j-n|>|n|/2$: In this case

$$2|j-n|>|n|,\quad \langle \ell+n\rangle^{\delta}\leqslant\langle \ell+k\rangle^{\delta}\langle -k+n\rangle^{\delta},$$

hence

$$H_n(\ell,k,j)\leqslant\frac{2^{\delta}}{\langle k-n\rangle^{1-\delta}\langle j-n\rangle^{1-\delta}\langle k+j\rangle^{\delta}\langle j+n\rangle^{\delta}}\leqslant\frac{2^{\delta}}{\langle k-n\rangle^{1-\delta}\langle k+j\rangle^{\delta}\langle j+n\rangle^{\delta}}.$$

Hence in both cases we obtain the same estimate. Define $\widetilde{e}(\ell+n)\equiv w(\ell+n)e(\ell+n)$ by

$$\widetilde{e}(\ell+n):=\sum_{k,j}\left(\frac{1}{\langle k-n\rangle\langle j+n\rangle^{\delta}\langle k+j\rangle^{\delta}}\right)\widetilde{a}(k+\ell)\widetilde{b}(\ell+j)\widetilde{d}(j+n).$$

Then we have

$$\langle n\rangle^{\delta+\alpha}w(n+\ell)\Psi_n(a,b,d_n)(\ell+n)\leqslant w(\ell+n)e(\ell+n)$$

and

$$\widetilde{e}(\ell)=\left(\widetilde{a}*\frac{1}{\langle k\rangle^{1-\delta}}\left(\frac{J\widetilde{b}}{\langle k\rangle^{\delta}}*\frac{\widetilde{d}}{\langle k\rangle^{\delta}}\right)\right)(\ell).$$

By Corollary B.2(ii),

$$\|\widetilde{e}\|_{\ell^2}\leqslant C\|a\|_w\|b\|_w\|d\|_w$$

for some constant $1\leqslant C\equiv C_\delta<\infty$. □

It remains to establish the auxilary results used in the proof of Lemma 2.8. First we need the following

Lemma B.1. *Let* $0<\eta\leqslant 1$. *Then*

(i) $\left\|\frac{a}{\langle k\rangle^{\eta}}\right\|_{\ell^p}\leqslant C_{p,\eta}\|a\|_{\ell^2}\quad\forall a\in\ell^2$ *and* $p>2/(2\eta+1)$;

(ii) $\left\|\frac{a}{\langle k\rangle^\eta}\right\|_{\ell^1} \leqslant C_{q,\eta}\|a\|_{\ell^q} \quad \forall a \in \ell^q$ *with* $1 \leqslant q < 1/(1-\eta)$.

Proof. (i) follows from Hölder's inequality with $\alpha = 2/p$ and $\beta = 2/(2-p)$,

$$\left(\sum_k \left(\frac{a(k)}{\langle k\rangle^\eta}\right)^p\right)^{1/p} = \left(\sum_k a(k)^p \frac{1}{\langle k\rangle^{\eta p}}\right)^{1/p} \leqslant \left(\sum_k |a(k)|^2\right)^{1/2}\left(\sum_k \frac{1}{\langle k\rangle^{\eta p\beta}}\right)^{1/\beta p},$$

where $\eta p\beta = \eta p \cdot 2/(2-p) > 1$ or $2\eta p > 2-p$ as $(2\eta+1)p > 2$ by assumption.

(ii) follows from Hölder's inequality with $\alpha = q$ and $1/\beta = 1 - 1/q = (q-1)/q$

$$\sum_k \frac{|a(k)|}{\langle k\rangle^\eta} \leqslant \left(\sum_k a(k)^q\right)^{1/q} \left(\sum_k \left(\frac{1}{\langle k\rangle^\eta}\right)^{q/(q-1)}\right)^{(q-1)/q},$$

where $\eta q/(q-1) > 1$ or $\eta q > q-1$ as $1 > (1-\eta)q$ by assumption. □

Recall Young's inequality

$$\|u * v\|_q \leqslant C_{r,p,q}\|u\|_p\|v\|_r,$$

where $r, p, q \geqslant 1$ with $1/p + 1/r = 1 + 1/q$.

Corollary B.2. *Let* $0 < \delta \leqslant 1/2$

(i) $\left\|\frac{1}{\langle k\rangle}\left(\frac{a}{\langle k\rangle^\delta} * b\right)\right\|_{\ell^1} \leqslant C_\delta\|a\|_{\ell^2}\|b\|_{\ell^2} \quad \forall a, b \in \ell^2;$

(ii) $\left\|\frac{1}{\langle k\rangle^{1-\delta}}\left(\frac{a}{\langle k\rangle^\delta} * \frac{b}{\langle k\rangle^\delta}\right)\right\|_{\ell^1} \leqslant C_\delta\|a\|_{\ell^2}\|b\|_{\ell^2} \quad \forall a, b \in \ell^2.$

Proof. (i) Let $1/p := 1/2 + \delta/2$ and $1/q := \delta/2$. Then $1/p + 1/2 = 1 + \delta/2 = 1 + 1/q$ and hence by Young's inequality

$$\left\|\frac{a}{\langle k\rangle^\delta} * b\right\|_{\ell^q} \leqslant C\left\|\frac{a}{\langle k\rangle^\delta}\right\|_{\ell^p}\|b\|_{\ell^2}.$$

As $p = 2/(1+\delta) > 2/(1+2\delta)$, Lemma B.1(i) can be applied,

$$\left\|\frac{a}{\langle k\rangle^\delta}\right\|_{\ell^p} \leqslant C\|a\|_{\ell^2},$$

and as $q = 2/\delta < \infty$, Lemma B.1(ii) gives

$$\left\|\frac{1}{\langle k\rangle}\left(\frac{a}{\langle k\rangle^\delta} * b\right)\right\|_{\ell^1} \leqslant C\|a\|_{\ell^2}\|b\|_{\ell^2}$$

as claimed.

(ii) By Lemma B.1, for $1/p := 1/2 + 2\delta/3$ $(< 1/2 + \delta)$

$$\left\| \frac{a}{\langle k \rangle^{\delta}} \right\|_{\ell^p} \leqslant C\|a\|_{\ell^2}.$$

By Young's inequality with $2 \cdot 1/p = 1 + 1/q$ or $1/q = 4\delta/3 < 1$ (as $0 < \delta \leqslant 1/2$)

$$\left\| \frac{a}{\langle k \rangle^{\delta}} * \frac{b}{\langle k \rangle^{\delta}} \right\|_{\ell^q} \leqslant C\|a\|_{\ell^2}\|b\|_{\ell^2}.$$

By Lemma B.1(ii) with $\eta = 1 - \delta$ (hence $1 \leqslant q = 3/(4\delta) < 1/\delta = 1/(1-\eta)$)

$$\left\| \frac{1}{\langle k \rangle^{1-\delta}} \left(\frac{a}{\langle k \rangle^{\delta}} * \frac{b}{\langle k \rangle^{\delta}} \right) \right\|_{\ell^1} \leqslant C\|a\|_{\ell^2}\|b\|_{\ell^2},$$

where $1 \leqslant C \equiv C_\delta < \infty$ depends only on δ. $\square$

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